美国数学会经典影印系列
AMS
AMERICAN MATHEMATICAL SOCIETY

Lectures on the Orbit Method

轨道法讲义

A. A. Kirillov

高等教育出版社

美国数学会经典影印系列

AMS
AMERICAN MATHEMATICAL SOCIETY

Lectures on the Orbit Method

轨道法讲义

A. A. Kirillov

高等教育出版社·北京

图字：01-2016-2512 号

Lectures on the Orbit Method, by A. A. Kirillov, first published by
the American Mathematical Society.
Copyright © 2004 by the American Mathematical Society. All rights reserved.
This present reprint edition is published by Higher Education Press Limited Company under authority of the American Mathematical Society and is published under license.
Special Edition for People's Republic of China Distribution Only. This edition has been authorized by the American Mathematical Society for sale in People's Republic of China only, and is not for export therefrom.

本书最初由美国数学会于 2004 年出版，原书名为 *Lectures on the Orbit Method*，
作者为 A. A. Kirillov。美国数学会保留原书所有版权。
原书版权声明：Copyright © 2004 by the American Mathematical Society。
本影印版由高等教育出版社有限公司经美国数学会独家授权出版。
本版只限于中华人民共和国境内发行。本版经由美国数学会授权仅在中华人民共和国境内销售，不得出口。

轨道法讲义
Guidaofa Jiangyi

图书在版编目（CIP）数据

轨道法讲义 = Lectures on the Orbit Method：英文 /（美）基里洛夫（A. A. Kirillov）著 . —影印本. —北京 : 高等教育出版社，2017.1
ISBN 978-7-04-046910-3
Ⅰ. ①轨… Ⅱ. ①基… Ⅲ. ①李群—英文 ②表示（数学）—英文 Ⅳ. ①O152
中国版本图书馆 CIP 数据核字 (2016) 第 281977 号

策划编辑 赵天夫　　责任编辑 赵天夫
封面设计 张申申　　责任印制 韩 刚

出版发行 高等教育出版社
社址 北京市西城区德外大街 4 号
邮政编码 100120
购书热线 010-58581118
咨询电话 400-810-0598
网址 http://www.hep.edu.cn
http://www.hep.com.cn
网上订购 http://www.hepmall.com.cn
http://www.hepmall.com
http://www.hepmall.cn
印刷 涿州市星河印刷有限公司

开本 787mm × 1092mm 1/16
印张 27.25
字数 610 千字
版次 2017 年 1 月第 1 版
印次 2017 年 1 月第 1 次印刷
定价 169.00 元

本书如有缺页、倒页、脱页等质量问题，
请到所购图书销售部门联系调换
版权所有 侵权必究
[物 料 号 46910-00]

To Kirill, Vanya, Lena, and Andrei

Preface

The goal of these lectures is to describe the essence of the orbit method for non-experts and to attract the younger generation of mathematicians to some old and still unsolved problems in representation theory where I believe the orbit method could help.

It is said that to become a scientist is the same as to catch a train at full speed. Indeed, while you are learning well-known facts and theories, many new important achievements happen. So, you are always behind the present state of the science. The only way to overcome this obstacle is to "jump", that is, to learn very quickly and thoroughly some relatively small domain, and have only a general idea about all the rest.

So, in my exposition I deliberately skip many details that are not absolutely necessary for understanding the main facts and ideas. The most persistent readers can try to reconstruct these details using other sources. I hope, however, that for the majority of users the book will be sufficiently self-contained.

The level of exposition is different in different chapters so that both experts and beginners can find something interesting and useful for them. Some of this material is contained in my book [**Ki2**] and in the surveys [**Ki5**], [**Ki6**], and [**Ki9**]. But a systematic and reasonably self-contained exposition of the orbit method is given here for the first time.

I wrote this book simultaneously in English and in Russian. For several reasons the English edition appears later than the Russian one and differs from it in the organization of material.

Sergei Gelfand was the initiator of the publication of this book and pushed me hard to finish it in time.

Craig Jackson read the English version of the book and made many useful corrections and remarks.

The final part of the work on the book was done during my visits to the Institut des Hautes Études Scientifiques (Bures-sur-Yvette, France) and the Max Planck Institute of Mathematics (Bonn, Germany). I am very grateful to both institutions for their hospitality.

In conclusion I want to thank my teachers, friends, colleagues, and especially my students, from whom I learned so much.

Introduction

The idea behind the orbit method is to unite harmonic analysis with symplectic geometry. This can be considered as a part of the more general idea of the unification of mathematics and physics.

In fact, this is a *post factum* formulation. Historically, the orbit method was proposed in [**Ki1**] for the description of the unitary dual (i.e. the set of equivalence classes of unitary irreducible representations) of nilpotent Lie groups. It turned out that the method not only solves this problem but also gives simple and visual solutions to all other principal questions in representation theory: topological structure of the unitary dual, the explicit description of the restriction and induction functors, the formulae for generalized and infinitesimal characters, the computation of the Plancherel measure, etc.

Moreover, the answers make sense for general Lie groups and even beyond, although sometimes with more or less evident corrections. I already mentioned in [**Ki1**] the possible applications of the orbit method to other types of Lie groups, but the realization of this program has taken a long time and is still not accomplished despite the efforts of many authors.

I cannot mention here all those who contributed to the development of the orbit method, nor give a complete bibliography: Mathematical Reviews now contains hundreds of papers where coadjoint orbits are mentioned and thousands of papers on geometric quantization (which is the physical counterpart of the orbit method). But I certainly ought to mention the outstanding role of Bertram Kostant and Michel Duflo.

As usual, the faults of the method are the continuations of its advantages. I quote briefly the most important ones.

MERITS VERSUS DEMERITS

Merits	Demerits
1. Universality: the method works for Lie groups of any type over any field.	1. The recipes are not accurately and precisely developed.
2. The rules are visual, and are easy to memorize and illustrate by a picture.	2. Sometimes they are wrong and need corrections or modifications.
3. The method explains some facts which otherwise look mysterious.	3. It could be difficult to transform this explanation into a rigorous proof.
4. It provides a great amount of symplectic manifolds and Poisson commuting families of functions.	4. Most of the completely integrable dynamical systems were discovered earlier by other methods.
5. The method introduces new fundamental notions: coadjoint orbit and moment map.	5. The description of coadjoint orbits and their structures is sometimes not an easy problem.

For the reader's convenience we formulate the ideology of the orbit method here in the form of a "User's Guide" where practical instructions are given as to how to get answers to ten basic questions in representation theory.

These simple rules are applicable literally for all connected and simply connected nilpotent groups. For groups of general type we formulate the "ten amendments" to these rules in the main text of the book.

Throughout the User's Guide we use the following notation:

G – a connected simply connected Lie group;

$H \subset G$ – a closed connected subgroup;

$\mathfrak{g}$, $\mathfrak{h}$ – Lie algebras of G, H, respectively;

$\mathfrak{g}^*$, $\mathfrak{h}^*$ – the dual spaces to $\mathfrak{g}$, $\mathfrak{h}$, respectively;

$p\colon \mathfrak{g}^* \to \mathfrak{h}^*$ – the canonical projection;

σ – the canonical 2-form (symplectic structure) on a coadjoint orbit;

π_Ω – the unirrep of G corresponding to the orbit $\Omega \subset \mathfrak{g}^*$;

$\rho_{F,H}$ – the 1-dimensional unirrep of H given by $\rho_{F,H}(\exp X) = e^{2\pi i\langle F,\, X\rangle}$;

P_A – the G-invariant polynomial on $\mathfrak{g}^*$ related to $A \in Z(\mathfrak{g})$, the center of $U(\mathfrak{g})$.

For other notation, when it is not self-explanatory, the reader must consult the Index and look for definitions given in the main text or in the Appendices.

USER'S GUIDE

What you want	What you have to do
1. Describe the unitary dual $\widehat{G}$ as a topological space.	Take the space $\mathcal{O}(G)$ of coadjoint orbits with the quotient topology.
2. Construct the unirrep π_Ω associated to the orbit $\Omega \in \mathfrak{g}^*$.	Choose a point $F \in \Omega$, take a subalgebra $\mathfrak{h}$ of maximal dimension subordinate to F, and put $\pi_\Omega = \operatorname{Ind}_H^G \rho_{F,H}$.
3. Describe the spectrum of $\operatorname{Res}_H^G \pi_\Omega$.	Take the projection $p(\Omega)$ and split it into H-orbits.
4. Describe the spectrum of $\operatorname{Ind}_H^G \pi_\omega$.	Take the G-saturation of $p^{-1}(\omega)$ and split it into G-orbits.
5. Describe the spectrum of the tensor product $\pi_{\Omega_1} \otimes \pi_{\Omega_2}$.	Take the arithmetic sum $\Omega_1 + \Omega_2$ and split it into orbits.
6. Compute the generalized character of π_Ω.	$\operatorname{tr}\, \pi_\Omega(\exp X) = \int_\Omega e^{2\pi i\langle F,X\rangle + \sigma}$ or $\langle \chi_\Omega\ ,\ \varphi\rangle = \int_\Omega \widetilde{\varphi}\,(F)\, e^\sigma$.
7. Compute the infinitesimal character of π_Ω.	For $A \in Z(\mathfrak{g})$ take the value of $P_A \in Pol(\mathfrak{g}^*)^G$ on the orbit Ω.
8. What is the relation between π_Ω and $\pi_{-\Omega}$?	They are contragredient (dual) representations.
9. Find the functional dimension of π_Ω.	It is equal to $\frac{1}{2}\dim\Omega$.

10. Compute the Plancherel measure μ on $\widehat{G}$.	The measure on $\mathcal{O}(G)$ arising when the Lebesgue measure on $\mathfrak{g}^*$ is decomposed into canonical measures on coadjoint orbits.

These short instructions are developed in Chapter 3 and illustrated in the worked-out examples in the main text.

Finally, a technical remark. I am using the standard sign $\square$ to signal the end of a proof (or the absence of proof). I also use less standard notation:

$\diamondsuit$ – the end of an example;

$\heartsuit$ – the end of a remark;

$\clubsuit$ – the end of an exercise;

$\spadesuit$ – the end of a warning about a possible mistake or misunderstanding.

The most difficult exercises and parts of the text are marked by an asterisk (*).

Contents

Chapter 1

Geometry of Coadjoint Orbits

We start our book with the study of coadjoint orbits. This notion is the main ingredient of the orbit method. It is also the most important new mathematical object that has been brought into consideration in connection with the orbit method.

1. Basic definitions

By a **coadjoint orbit** we mean an orbit of a Lie group G in the space $\mathfrak{g}^*$ dual to $\mathfrak{g} = \mathrm{Lie}(G)$. The group G acts on $\mathfrak{g}^*$ via the coadjoint representation, dual to the adjoint one (see the definition below and also Appendix III.1.1).

In this chapter we consider the geometry of coadjoint orbits and discuss the problem of their classification.

1.1. Coadjoint representation.

Let G be a Lie group. It is useful to have in mind the particular case when G is a matrix group, i.e. a subgroup and at the same time a smooth submanifold of $GL(n, \mathbb{R})$.

Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the tangent space $T_e(G)$ to G at the unit point e. The group G acts on itself by **inner automorphisms**: $A(g)\colon x \mapsto g x g^{-1}$. The point e is a fixed point of this action, so we can define the derived map $\big(A(g)\big)_*(e)\colon \mathfrak{g} \to \mathfrak{g}$. This map is usually denoted by $\mathrm{Ad}(g)$.

The map $g \mapsto \mathrm{Ad}(g)$ is called the **adjoint representation** of G. In the case of a matrix group G the Lie algebra $\mathfrak{g}$ is a subspace of $\mathrm{Mat}_n(\mathbb{R})$ and

the adjoint representation is simply the matrix conjugation:

$$\text{(1)} \qquad \operatorname{Ad}(g)X = g \cdot X \cdot g^{-1}, \quad X \in \mathfrak{g}, \quad g \in G.$$

The same formula holds in the general case if we accept the matrix notation introduced in Appendix III.1.1.

Consider now the vector space dual to $\mathfrak{g}$. We shall denote it by $\mathfrak{g}^*$. Recall that for any linear representation (π, V) of a group G one can define a dual representation (π^*, V^*) in the dual space V^*:

$$\pi^*(g) := \pi(g^{-1})^*$$

where the asterisk in the right-hand side means the dual operator in V^* defined by

$$\langle A^* f, v\rangle := \langle f, Av\rangle \qquad \text{for any } v \in V, \ f \in V^*.$$

In particular, we have a representation of a Lie group G in $\mathfrak{g}^*$ that is dual to the adjoint representation in $\mathfrak{g}$. This representation is called **coadjoint**.

Since this notion is very important, and also for brevity, we use the special notation[1] $K(g)$ for it instead of the full notation $\operatorname{Ad}^*(g) = \operatorname{Ad}(g^{-1})^*$. So, by definition,

$$\text{(2)} \qquad \langle K(g)F, X\rangle = \langle F, \operatorname{Ad}(g^{-1})X\rangle$$

where $X \in \mathfrak{g}$, $F \in \mathfrak{g}^*$, and by $\langle F, X\rangle$ we denote the value of a linear functional F on a vector X.

For matrix groups we can use the fact that the space $\operatorname{Mat}_n(\mathbb{R})$ has a bilinear form

$$\text{(3)} \qquad (A, B) = \operatorname{tr}(AB),$$

which is non-degenerate and invariant under conjugation. So, the space $\mathfrak{g}^*$, dual to the subspace $\mathfrak{g} \subset \operatorname{Mat}_n(\mathbb{R})$, can be identified with the quotient space $\operatorname{Mat}_n(\mathbb{R})/\mathfrak{g}^\perp$. Here the sign $^\perp$ means the orthogonal complement with respect to the form (,):

$$\mathfrak{g}^\perp = \{A \in \operatorname{Mat}_n(\mathbb{R}) \mid (A, B) = 0 \quad \text{for all } B \in \mathfrak{g}\}.$$

In practice the quotient space is often identified with a subspace $V \subset \operatorname{Mat}_n(\mathbb{R})$ that is transversal to $\mathfrak{g}^\perp$ and has the complementary dimension. Therefore, we can write $\operatorname{Mat}_n(\mathbb{R}) = V \oplus \mathfrak{g}^\perp$. Let p_V be the projection of

[1] In Russian the word "coadjoint" starts with k.

$\mathrm{Mat}_n(\mathbb{R})$ onto V parallel to $\mathfrak{g}^\perp$. Then the coadjoint representation K can be written in a simple form

$$K(g)\colon F \mapsto p_V\,(gFg^{-1}). \tag{4}$$

Remark 1. If we could choose V invariant under $\mathrm{Ad}\,(G)$ (which we can always assume for $\mathfrak{g}$ semisimple or reductive), then we can omit the projection p_V in (4). ♡

Example 1. Denote by G the group of all (non-strictly) upper triangular matrices $g \in GL(n, \mathbb{R})$, i.e. such that $g_{ij} = 0$ for $i > j$. Then the Lie algebra $\mathfrak{g}$ consists of all upper triangular matrices from $\mathrm{Mat}_n(\mathbb{R})$. The space $\mathfrak{g}^\perp$ is the space of strictly upper triangular matrices X satisfying the condition $x_{ij} = 0$ for $i \geq j$.

We can take for V the space of all lower triangular matrices.

The projection p_V in this case sends any matrix to its "lower part" (i.e. replaces all entries above the main diagonal by zeros). Hence, the coadjoint representation takes the form

$$K(g): \quad F \mapsto (g\,Fg^{-1})_{\text{lower part}}.$$

Although this example has been known for a long time and has been thoroughly studied by many authors, we still do not know how to classify the coadjoint orbits for general n. ♢

Example 2. Let $G = SO(n, \mathbb{R})$. Then $\mathfrak{g}$ consists of all skew-symmetric matrices $X = -X^t$ from $\mathrm{Mat}_n(\mathbb{R})$.

Here we can put $V = \mathfrak{g}$ and omit the projection p_V in (4) (cf. Remark 1):

$$K(g)X = g \cdot X \cdot g^{-1}.$$

Thus, the coadjoint representation is equivalent to the adjoint one and coincides with the standard action of the orthogonal group on the space of antisymmetric bilinear forms. It is well known that a coadjoint orbit passing through X is determined by the spectrum of X, which can be any multiset in $i\mathbb{R}$, symmetric with respect to the complex conjugation. Another convenient set of parameters labelling the orbits is the collection of real numbers $\{\mathrm{tr}\,X^2,\ \mathrm{tr}\,X^4,\ \ldots,\ \mathrm{tr}\,X^{2k}\}$ where $k = [\frac{n}{2}]$. ♢

We also give the formula for the infinitesimal version of the coadjoint action, i.e. for the corresponding representation K_* of the Lie algebra $\mathfrak{g}$ in $\mathfrak{g}^*$:

$$\langle K_*(X)F,\ Y\rangle = \langle F,\ -\mathrm{ad}(X)\ Y\rangle = \langle F,\ [Y, X]\rangle. \tag{5}$$

For matrix groups it takes the form

$$(5') \qquad K_*(X)F = p_V([X, F]) \quad \text{for } X \in \mathfrak{g},\ F \in V \simeq \mathfrak{g}^*.$$

Remark 2. The notions of coadjoint representation and coadjoint orbit can be defined beyond the realm of Lie groups in the ordinary sense.

Three particular cases are of special interest: infinite-dimensional groups, algebraic groups over arbitrary fields and quantum groups (which are not groups at all). In all three cases the ideology of the orbit method seems to be very useful and often suggests the right formulations of important results.

We discuss below some examples of such results, although this subject is outside the main scope of the book. ♡

1.2. Canonical form σ_Ω.

One feature of coadjoint orbits is eye-catching when you consider a few examples: they always have an even dimension. This is not accidental, but has a deep geometric reason.

All coadjoint orbits are symplectic manifolds. Moreover, each coadjoint orbit possesses a canonical G-invariant symplectic structure. This means that on each orbit $\Omega \subset \mathfrak{g}^*$ there is a canonically defined closed non-degenerate G-invariant differential 2-form σ_Ω.

In the next sections we give several explanations of this phenomenon and here just give the definition of σ_Ω.

We use the fact that a G-invariant differential form ω on a homogeneous G-manifold $M = G/H$ is uniquely determined by its value at the initial point m_0 and this value can be any H-invariant antisymmetric polylinear form on the tangent space $T_{m_0}M$.

Thus, to define σ_Ω it is enough to specify its value at some point $F \in \Omega$, which must be an antisymmetric bilinear form on $T_F\Omega$ invariant under the action of the group $Stab\,F$, the stabilizer of F.

Let $stab(F)$ be the Lie algebra of $Stab(F)$. We can consider the group G as a fiber bundle over the base $\Omega \simeq G/Stab(F)$ with projection

$$p_F:\ G \to \Omega, \qquad p_F(g) = K(g)F.$$

It is clear that the fiber above the point F is exactly $Stab(F)$. Consider the exact sequence of vector spaces

$$0 \to stab(F) \hookrightarrow \mathfrak{g} \xrightarrow{(p_F)_*} T_F(\Omega) \to 0$$

that comes from the above interpretation of G as a fiber bundle over Ω. It allows us to identify the tangent space $T_F(\Omega)$ with the quotient $\mathfrak{g}/stab(F)$.

Now we introduce the antisymmetric bilinear form B_F on $\mathfrak{g}$ by the formula

$$B_F(X,Y) = \langle F,\ [X,Y]\rangle. \tag{6}$$

Lemma 1. *The kernel of B_F is exactly $stab(F)$.*

Proof.

$$\begin{aligned}\ker B_F &= \{X \in \mathfrak{g} \mid B_F(X,Y) = 0\ \forall\, Y \in \mathfrak{g}\}\\ &= \{X \in \mathfrak{g} \mid \langle K_*(X)F,\ Y\rangle = 0\ \forall\, Y \in \mathfrak{g}\}\\ &= \{X \in \mathfrak{g} \mid K_*(X)F = 0\} = stab(F).\end{aligned}$$

□

Lemma 2. *The form B_F is invariant under $Stab(F)$.*

Proof.

$$\langle F,\ [\mathrm{Ad}h X, \mathrm{Ad}h Y]\rangle = \langle F, \mathrm{Ad}h[X,Y]\rangle = \langle K(h^{-1})F, [X,\ Y]\rangle = \langle F, [X,Y]\rangle$$

for any $h \in Stab(F)$. □

Now we are ready to introduce

Definition 1. Let Ω be a coadjoint orbit in $\mathfrak{g}^*$. We define the differential 2-form σ_Ω on Ω by

$$\sigma_\Omega(F)(K_*(X)F,\ K_*(Y)F) = B_F(X,\ Y). \tag{7}$$

The correctness of this definition, as well as the non-degeneracy and G-invariance of the constructed form follows immediately from the discussion above.

2. Symplectic structure on coadjoint orbits

The goal of this section is to prove

Theorem 1. *The form σ_Ω is closed, hence defines on Ω a G-invariant symplectic structure.*

There exist several proofs of this theorem that use quite different approaches. This can be considered as circumstantial evidence of the depth and importance of the theorem. Three and a half of these proofs are presented below.

2.1. The first (original) approach.

We use the explicit formula (19) in Appendix II.2.3 for the differential of a 2-form:

$$d\sigma(\xi,\,\eta,\,\zeta) = \circlearrowleft \xi\sigma(\eta,\,\zeta) - \circlearrowleft \sigma([\xi,\,\eta],\,\zeta)$$

where the sign $\circlearrowleft$ denotes the summation over cyclic permutations of $\xi,\,\eta,\,\zeta$.

Let $\xi,\,\eta,\,\zeta$ be the vector fields on Ω which correspond to elements $X,\,Y,\,Z$ of the Lie algebra $\mathfrak{g}$.[2] Then $\xi(F) = K_*(X)F,\ \eta(F) = K_*(Y)F,\ \zeta(F) = K_*(Z)F$, and we obtain

$$\sigma(\eta,\,\zeta) = \langle F,\,[Y,\,Z]\rangle, \qquad \xi\sigma(\eta,\,\zeta) = \langle K_*(X)F,\,[Y,\,Z]\rangle = -\langle F,\,[X,\,[Y,\,Z]]\rangle,$$
$$[\xi,\,\eta] = -K_*([X,\,Y])F, \qquad \sigma([\xi,\,\eta],\,\zeta) = -\langle F,\,[[X,\,Y],\,Z]\rangle.$$

Therefore,

$$d\sigma(\xi,\,\eta,\,\zeta) = 2 \circlearrowleft \langle F,\,[X,\,[Y,\,Z]]\rangle = 0 \quad \text{via the Jacobi identity.}$$

Since G acts transitively on Ω, the vectors $K_*(X)F,\ X \in \mathfrak{g}$, span the whole tangent space $T_F\Omega$. Thus, $d\sigma = 0$.

This proof of Theorem 1, being short enough, can be however not quite satisfactory for a geometric-minded reader. So, we give a variant of it which is based on more geometric observations. This variant of the proof is also in accordance with the general metamathematical homology principle mentioned in Remark 1 in Appendix I.3.

Consider again the fibration $p_F : G \to \Omega$ and introduce the form $\Sigma_F := p_F^*(\sigma)$ on the group G. By the very construction, Σ_F is a left-invariant 2-form on G with initial value $\Sigma_F(e) = B_F$. We intend to show that this form is not only closed but exact.

To see this, we shall use the so-called **Maurer-Cartan form** Θ. By definition, it is a $\mathfrak{g}$-valued left-invariant 1-form on G defined by the condition $\Theta(e)(X) = X$. Since the left action of G on itself is **simply transitive** (i.e. there is exactly one left shift which sends a given point g_1 to another given point g_2), to define a left-invariant form on G we only need to specify arbitrarily its value at one point. The explicit formula for Θ in matrix notations is

$$\Theta(g)(X) = g^{-1}\cdot X \quad \text{for } X \in T_gG. \tag{8}$$

Often, especially in physics papers, this form is denoted by $g^{-1}dg$ because for any smooth curve $g = g(t)$ we have $g^*\Theta = g(t)^{-1}\dot{g}(t)dt = g^{-1}dg$.

[2] Recall that for left G-manifolds the vector field ξ corresponding to $X \in \mathfrak{g}$ is defined by $\xi(x) = \frac{d}{dt}(\exp tX)\cdot x\,\big|_{t=0}$. The map $X \mapsto \xi$ is an antihomomorphism of $\mathfrak{g}$ to *Vect* M.

Proposition 1. *The 2-form $\Sigma_F = p_F^*(\sigma)$ on G is the exterior derivative of the left-invariant real-valued 1-form θ_F given by*

$$\theta_F = -\langle F,\, \Theta \rangle. \tag{9}$$

Proof. We shall use the formula for the exterior derivative of a 1-form (see (19) from Appendix II.2.3):

$$d\theta(\xi,\, \eta) = \xi\theta(\eta) - \eta\theta(\xi) - \theta([\xi,\, \eta]).$$

Let $\widetilde{X}$ and $\widetilde{Y}$ be left-invariant vector fields on G (see the fourth definition of a Lie algebra in Appendix III.1.3). Putting $\theta = \theta_F,\ \xi = \widetilde{X},\ \eta = \widetilde{Y}$, we get

$$d\theta_F(\widetilde{X},\, \widetilde{Y}) = \widetilde{X}\theta_F(\widetilde{Y}) - \widetilde{Y}\theta_F(\widetilde{X}) - \theta_F([\widetilde{X},\, \widetilde{Y}]).$$

The first and second terms in the right-hand side vanish because $\theta_F(\widetilde{X})$ and $\theta_F(\widetilde{Y})$ are constant functions. We can rewrite the last term as

$$-\theta_F([\widetilde{X},\, \widetilde{Y}]) = -\theta_F(\widetilde{[X,\, Y]}) = \langle F,\, [X,\, Y] \rangle = p_F^*(\sigma)(\widetilde{X},\, \widetilde{Y}).$$

□

Now we return to the form σ. Since p_F is a submersion, the linear map $(p_F)_*$ is surjective. Therefore, the dual map p_F^* is injective. But $p_F^* d\sigma = dp_F^*(\sigma) = d\,\Sigma_F = d^2\theta_F = 0$. Hence, σ is closed.

Note that in general θ_F cannot be written as $p_F^*(\phi)$ for some 1-form ϕ on Ω, so we cannot claim that σ is exact (and actually it is not in general).

2.2. The second (Poisson) approach.

We now discuss another way to introduce the canonical symplectic structure on coadjoint orbits. It is based on the notion of Poisson manifold (see Appendix II.3.2).

Consider the real n-dimensional vector space V and, making an exception to the general rules, denote the coordinates $(X_1, \ldots, X_n)$ on V using lower indices. Let c be a bivector field on V with linear coefficients:

$$c = c_{ij}^k X_k\, \partial^i \otimes \partial^j \tag{10}$$

where $\partial^i = \partial/\partial_{X_i}$, $\partial^j = \partial/\partial_{X_j}$, and $c_{ij}^k = -c_{ji}^k$.

Lemma 3. *The bivector* (10) *defines a Poisson structure on V if and only if the coefficients c_{ij}^k form a collection of structure constants for some Lie algebra $\mathfrak{g}$.*

Proof. Consider the bracket operation defined by c:

$$\{f_1, f_2\} = c_{ij}^k X_k \frac{\partial f_1}{\partial X_i} \frac{\partial f_2}{\partial X_j}. \tag{11}$$

Since the bivector c has linear coefficients, the space V^* of linear functions on V is closed under this operation.

If c defines a Poisson structure on V, then V^* is a Lie subalgebra in $C^\infty(V)$ that we denote by $\mathfrak{g}$. Therefore, V itself can be identified with the dual space $\mathfrak{g}^*$, which justifies the labelling of coordinates by lower indices.

In the natural basis in V^* formed by coordinates $(X_1, \ldots, X_n)$ the coefficients c_{ij}^k are precisely the structure constants of $\mathfrak{g}$:

$$\{X_i, X_j\} = \sum_k c_{ij}^k X_k.$$

Conversely, if c_{ij}^k are structure constants of a Lie algebra $\mathfrak{g}$, then the brackets (11) satisfy the Jacobi identity for any linear functions f_1, f_2, f_3. But this identity involves only the first partial derivatives of f_i. Therefore, it is true for all functions. □

Remark 3. The existence of a Poisson bracket on $\mathfrak{g}^*$ was already known to Sophus Lie in 1890, as was pointed out recently by A. Weinstein. It seems that Lie made no use of it. F. A. Berezin rediscovered this bracket in 1967 in connection with his study of universal enveloping algebras [**Be1**]. The relation of this fact to coadjoint orbits was apparently first noted in [**Ki4**]. ♡

This relation can be formulated as follows.

Theorem 2. *The symplectic leaves of the Poisson manifold* $(\mathfrak{g}^*, c)$ *are exactly the coadjoint orbits.*

Proof. Let L_F be the leaf that contains the point $F \in \mathfrak{g}^*$. The tangent space to L_F at F by definition (see Theorem 6 in Appendix II.3.2) is spanned by vectors $v_i = c_{ij}^k X_k \partial^j$. But v_i is exactly the value at F of the vector field on $\mathfrak{g}^*$ corresponding to the infinitesimal coadjoint action of $X_i \in \mathfrak{g}$. Therefore, the coadjoint orbit Ω_F and the leaf L_F have the same tangent space. Since it is true for every point $F \in \mathfrak{g}^*$, we get $L_F = \Omega_F$. □

Theorem 2 gives an alternative approach to the construction of the canonical symplectic structure on coadjoint orbits.

2.3.* The third (symplectic reduction) approach.

We apply the symplectic reduction procedure described in Appendix II.3.2 to the special case when the symplectic manifold M is the cotangent bundle T^*G over a Lie group G. This case has its peculiarities.

First, the bundle T^*G is trivial. Namely, we shall use the left action of G on itself to make the identification $T^*(G) \simeq G \times \mathfrak{g}^*$. In matrix notation the covector $g \cdot F \in T^*_{g^{-1}}G$ corresponds to the pair $(g, F) \in G \times \mathfrak{g}^*$.[3]

Further, the set $T^*(G)$ is itself a group with respect to the law

$$(g_1, F_1)(g_2, F_2) = (g_1 g_2, K(g_2)^{-1} F_1 + F_2).$$

If we identify $(g, 0) \in T^*(G)$ with $g \in G$ and $(e, F) \in T^*(G)$ with $F \in \mathfrak{g}^*$, then $T^*(G)$ becomes a semidirect product $G \ltimes \mathfrak{g}^*$. So, we can write $(g, F) = g \cdot F$ both in matrix notation and in the sense of the group law in T^*G.

Note also the identity $g \cdot F \cdot g^{-1} = K(g)F$.

Since T^*G is a Lie group, the tangent bundle $T(T^*(G))$ is also trivial. We identify $T_{(g,F)}T^*G$ with $\operatorname{Lie}(T^*G) \simeq \mathfrak{g} \oplus \mathfrak{g}^*$ using the left shift and obtain

$$T(T^*(G)) \simeq (G \times \mathfrak{g}^*) \times (\mathfrak{g} \oplus \mathfrak{g}^*).$$

Theorem 3. *The canonical symplectic structure on $T^*(G)$ in the trivialization above is given by the bilinear form σ:*

$$\sigma_{(g,F)}(X_1 \oplus F_1, X_2 \oplus F_2) = \langle F_1, X_2 \rangle - \langle F_2, X_1 \rangle - \langle F, [X_1, X_2] \rangle. \tag{12}$$

Proof. Let us compute first the canonical 1-form θ on $T^*(G)$. For a tangent vector $v = (g, F; X, F') \in T_{(g,F)}T^*G$ the projection to T_gG equals $X \cdot g^{-1}$. Therefore, $\theta(v) = \langle g \cdot F, X \cdot g^{-1} \rangle = \langle F, X \rangle$.

Now we can compute σ as the exterior derivative of θ:

$$\sigma(\xi_1, \xi_2) = \xi_1 \theta(\xi_2) - \xi_2 \theta(\xi_1) - \theta([\xi_1, \xi_2])$$

for any vector fields ξ_1, ξ_2 on T^*G.

We choose ξ_1, ξ_2 as the left-invariant fields on the Lie group T^*G with initial values

$$\xi_1(e, 0) = (X_1, F_1), \qquad \xi_2(e, 0) = (X_2, F_2).$$

[3] Recall that $g \cdot F$ is defined by $\langle g \cdot F, \xi \rangle = \langle F, \xi \cdot g^{-1} \rangle$.

Then $[\xi_1, \xi_2](e, 0) = ([X_1, X_2], K(X_1)F_2 - K(X_2)F_1)$ and we also have

$$\xi_1\theta(\xi_2) = \xi_1\langle F, X_2\rangle = \langle F_1, X_2\rangle, \qquad \xi_2\theta(\xi_1) = \xi_2\langle F, X_1\rangle = \langle F_2, X_1\rangle,$$
$$\theta([\xi_1, \xi_2]) = \langle F, [X_1, X_2]\rangle.$$

The desired formula (12) follows from these computations. □

The group $G\times G$ acts on G by left and right shifts: $(g_1, g_2)\cdot g = g_1 g\, g_2^{-1}$. This action extends to a Hamiltonian action on T^*G. Actually, the extension is again given by left and right shifts on elements G viewed as elements of the group T^*G. Using the above identification we can write it in the form

$$(g_1, g_2)\cdot(g, F) = \big(g_1 g\, g_2^{-1}, K(g_2)F\big)\,. \tag{13}$$

Recall (see Appendix III.1.1) that to any $X \in \mathfrak{g}$ there correspond two vector fields on G:

– the infinitesimal right shift $\widetilde{X}$, which is a unique left-invariant field satisfying $\widetilde{X}(e) = X$, and

– the infinitesimal left shift $\widehat{X}$, which is a unique right-invariant field satisfying $\widehat{X}(e) = X$.

Exercise 1. Using the above identifications, write explicitly

a) the vector fields $\widetilde{X}$ and $\widehat{X}$ on G;

b) their Hamiltonian lifts $\tilde{X}^*$ and $\hat{X}^*$ on $T^*(G)$.

Answer:

$$\begin{aligned} &\text{a) } \widetilde{X}(g) = X; && \widehat{X}(g) = \operatorname{Ad} g^{-1}X;\\ &\text{b) } \widetilde{X}^*(g, F) = (X, 0), && \widehat{X}^*(g, F) = \big(\operatorname{Ad} g^{-1}X, K_*(X)F\big). \end{aligned} \tag{14}$$

Hint. Use formula (13).

To construct the reduction of T^*G with respect to the left, right, or two-sided action of G (see Appendix II.3.4), we have to know the moment map.

Lemma 4. *For the action of $G \times G$ on T^*G the moment map $\mu : T^*G \to \mathfrak{g}^* \oplus \mathfrak{g}^*$ is given by the formula*

$$\mu\,(g, F) = (F \oplus K(g)F). \tag{15}$$

Proof. The proof follows from Theorem 3 and the explicit formulae above for θ, $\widetilde{X}^*$, $\widehat{X}^*$. □

For the left action the moment map looks especially simple (because we are using the left trivialization): $\mu(g, F) = F$.

The fiber of the moment map over $F \in \mathfrak{g}^*$ is $G \times \{F\}$. So, the reduced manifold is $G/Stab(F) \simeq \Omega_F$. We omit the (rather tautological) verification of the equality $\sigma_0 = \sigma_\Omega$.

We could get the same result using the right action of G on T^*G. Indeed, here $\mu(g, F) = K(g)F$ and $\mu^{-1}(\Omega_F) = G \times \Omega_F$. Therefore, the reduced manifold is $(G \times \Omega_F)/G \simeq \Omega_F$.

2.4. Integrality condition.

In Appendix II.2.4 we explain how to integrate a differential k-form on M over an oriented smooth k-dimensional submanifold.

More generally, define a **real** (resp. **integral**) **singular k-cycle** in a manifold M as a linear combination $C = \sum_i c_i \cdot \varphi_i(M_i)$ of images of smooth k-manifolds M_i under smooth maps $\varphi_i : M_i \to M$ with real (resp. integer) coefficients c_i. Then we can define the integral of the form $\omega \in \Omega^k(M)$ over a singular k-cycle C as

$$\int_C \omega = \sum_i \left(c_i \cdot \int_{M_i} \varphi_i^*(\omega) \right).$$

It is known that the integral of a closed k-form over a singular k-dimensional cycle C depends only on the homology class $[C] \in H_{sing}(M)$.[4] Moreover, a k-form is exact iff its integral over any k-cycle vanishes.

For future use we make the following

Definition 2. A coadjoint orbit Ω is **integral** if the canonical form σ has the property

$$\int_C \sigma \in \mathbb{Z} \quad \text{for every integral singular 2-cycle } C \text{ in } \Omega. \tag{16}$$

In particular, it is true when C is any smooth 2-dimensional submanifold $S \subset \Omega$.

[4] We will not give the accurate definitions here of the groups $H_{sing}(M, \mathbb{R})$ and $H_{sing}(M, \mathbb{Z})$ of singular homologies of a manifold M. For the applications in representation theory the information given in this section is quite enough.

The integrality condition has important geometric and representation-theoretic interpretations. They are revealed by the following

Proposition 2. *Assume that G is a simply connected Lie group. The following are equivalent:*

(i) *$\Omega \subset \mathfrak{g}^*$ is integral.*

(ii) *There exists a G-equivariant complex line bundle over Ω with a G-invariant Hermitian connection ∇ such that*

$$curv\,(\nabla) = 2\pi i \sigma. \tag{17}$$

(iii) *For any $F \in \Omega$ there exists a unitary 1-dimensional representation χ of the connected Lie group $Stab^\circ(F)$ such that*

$$\chi(\exp X) = e^{2\pi i \langle F,\, X\rangle}. \tag{18}$$

Observe that condition (i) is automatically true for homotopically trivial (i.e. contractible) orbits. It is also true when the canonical form σ is exact.

Proof. (i)$\Longleftrightarrow$(ii). Let L be a complex line bundle over Ω. Choose a covering of Ω by open sets $\{U_\alpha\}_{\alpha\in A}$ such that for any $\alpha \in A$ there exists a non-vanishing section s_α of L over U_α. Then we can specify a section s by the collection of functions $f_\alpha \in \mathcal{A}(U_\alpha)$ given by

$$s\,\big|_{U_\alpha} = f_\alpha \cdot s_\alpha.$$

A connection ∇ in a line bundle L is given by a family of differential 1-forms θ_α. Namely, define θ_α by $\nabla_v s_\alpha = \theta_\alpha(v)\cdot s_\alpha$ for any $v \in Vect\,(U_\alpha)$. In terms of these forms the covariant derivative is

$$\nabla_v = v + \theta_\alpha(v), \quad \text{i.e.} \quad s \leftrightarrow \{f_\alpha\} \Rightarrow \nabla_v s \leftrightarrow \{v f_\alpha + \theta_\alpha(v) f_\alpha\}.$$

The connection is Hermitian if a scalar product is defined in all fibers so that

$$v\cdot(s_1,\, s_2) = (\nabla_v s_1,\, s_2) + (s_1,\, \nabla_v s_2).$$

If we normalize s_α by the condition $(s_\alpha,\, s_\alpha) = 1$, this condition becomes $\theta_\alpha = -\overline{\theta_\alpha}$.

Let $c_{\alpha,\beta}$ be the transition functions, so that $f_\alpha = c_{\alpha,\beta} f_\beta$ on $U_{\alpha,\beta}$. Then the forms θ_α satisfy

$$\theta_\beta - \theta_\alpha = d\log c_{\alpha,\beta}.$$

Therefore, the form $d\theta_\alpha$ coincides with $d\theta_\beta$ on $U_{\alpha,\beta}$. Hence, the collection $\{d\theta_\alpha\}$ defines a single 2-form Θ on Ω. This form is called the **curvature**

form of the connection ∇ and is denoted $curv\,\nabla$. (In some books another normalization is used and the real form $\frac{1}{2\pi i}\Theta$ is called the curvature form.)

Exercise 2. Prove that

$$(curv\,\nabla)(v,\,w) = [\nabla_v,\,\nabla_w] - \nabla_{[v,\,w]}.$$

Hint. Use the formula for ∇_v above and the formula for $d\theta$. ♣

We will not bother the reader with the verification of the following fact:

Lemma 5. *The Čech cocycle corresponding to the* 2*-form* $curv\,\nabla$ *has the form*

$$c_{\alpha,\beta,\gamma} = \log\, c_{\alpha,\,\beta} + \log\, c_{\beta,\,\gamma} + \log\, c_{\gamma,\,\alpha}.$$

□

From this formula and from the relation $c_{\alpha,\,\beta}\cdot c_{\beta,\,\gamma}\cdot c_{\gamma,\,\alpha} = 1$ it follows immediately that the cohomology class of $curv\,\nabla$ belongs to $2\pi i\cdot H^2(\Omega,\,\mathbb{Z})$. Hence, the form $\sigma = \frac{1}{2\pi i} curv\,\nabla$ belongs to an integral cohomology class.

Conversely, let σ be a real 2-form on Ω, and let θ_α be the real antiderivative of σ on U_α. Then we can define the functions $c_{\alpha,\beta}$ on $U_{\alpha,\beta}$ so that $dc_{\alpha,\beta} = 2\pi i(\theta_\beta - \theta_\alpha)$. If σ has the property $[\sigma]\in H^2(\Omega,\,\mathbb{Z})$, these functions satisfy $c_{\alpha,\,\beta}\cdot c_{\beta,\,\gamma}\cdot c_{\gamma,\,\alpha} = 1$. Hence, they can be considered as the transition functions of some complex line bundle L over Ω. Since the θ_α are real, we can assume that $|c_{\alpha,\,\beta}| = 1$. Therefore, L admits a scalar product in fibers such that $\nabla_v = v + 2\pi i\theta_\alpha(v)$ is a Hermitian connection on L.

(ii)$\Longleftrightarrow$(iii). Since $stab(F) = \ker B_F$, the representation (18) is in fact a representation of an abelian Lie group $A = Stab^\circ(F)/[Stab^\circ(F),\, Stab^\circ(F)]$. This group has the form $\mathbb{T}^k\times\mathbb{R}^l$. The representation (18) corresponds to a unitary 1-dimensional representation of A iff it takes integer values on the k-dimensional lattice $\Lambda = \exp^{-1}(e)$.

Let $X\in\Lambda$ and denote by γ the loop in $Stab^\circ(F)$ which is the image of the segment $[0,\,X]\subset stab(F)$ under the exponential map.

Since G is supposed to be simply connected, the loop γ is the boundary of some 2-dimensional surface S in G, whose projection to Ω we denote by $p(S)$. The correspondence $[\gamma]\rightsquigarrow[S]$ is precisely the isomorphism $\pi_1(Stab(F))\simeq\pi_2(\Omega)$ (see formula (6) in Appendix I.2.3). Finally we have

$$\langle F,\,X\rangle = \int_\gamma \theta_F = \int_S d\theta_F = \int_{p(S)}\sigma.$$

□

3. Coadjoint invariant functions

3.1. General properties of invariants.

The ideology of the orbit method attaches significant importance to the classification problem for coadjoint orbits. The first step in the solution of this problem is to find all invariants of the group action. Polynomial and rational invariants are especially interesting.

We also need the notion of **relative invariants** defined as follows. Let X be a left G-set, and let λ be a multiplicative character of G. We say that a function f is a relative invariant of type λ if

$$f(g \cdot x) = \lambda(g) f(x) \quad \text{for all } g \in G.$$

It is known that for algebraic actions of complex algebraic groups on affine algebraic manifolds there are enough rational invariants to separate the orbits in the following sense.

Proposition 3 (see [**Bor, R**]). *The common level set of all rational invariants consists of a finite number of orbits, and a generic level is just one orbit.*

Moreover, each rational invariant can be written in the form $R = \frac{P}{Q}$ *where* P *and* Q *are relative polynomial invariants of the same type.* □

For real algebraic groups the common level sets of invariants can split into a finite number of connected components that are not separated by rational invariants. (Compare with the geometry of quadrics in a real affine plane. The two branches of a hyperbola is the most visual example.)

A useful scheme for the construction of invariants of a group G acting on a space X is the following. Suppose we can construct a subset $S \subset X$ which intersects all (or almost all) orbits in a single point. Any invariant function on X defines, by restriction, a function on S. Conversely, any function on S can be canonically extended to a G-invariant function defined on X (or almost everywhere on X).

If S is smooth (resp. algebraic, rational, etc.), then we get information about smooth (resp. algebraic, rational, etc.) invariants.

Warning. While the restriction to S usually preserves the nice properties of invariants, the extending procedure does not. For example, the extension of a polynomial function could only be rational (see Example 5 below). ♠

3.2. Examples.

Example 3. Let $G = GL(n, \mathbb{R})$ act on $X = \mathrm{Mat}_n(\mathbb{R})$ by conjugation. Let S be the affine subset consisting of matrices of the form

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_1 \end{pmatrix}.$$

One can check that S intersects almost all conjugacy classes in exactly one point. Geometrically this means that for almost all operators A on $\mathbb{R}^n$ there exists a **cyclic vector** ξ, i.e. such that the vectors $\xi, A\xi, A^2\xi, \ldots, A^{n-1}\xi$ form a basis in $\mathbb{R}^n$. It is clear that in this basis the matrix of A has the above form.

Using the section S we show that in this case polynomial invariants form an algebra $\mathbb{R}[c_1, c_2, \ldots, c_n]$. Indeed, every polynomial invariant restricted to S becomes a polynomial in $c_1, c_2, \ldots, c_n$. On the other hand, all the c_i's admit extensions as invariant polynomials on $\mathrm{Mat}_n(\mathbb{R})$. Namely, they coincide up to sign with the coefficients of the characteristic polynomial $P_A(\lambda) = \det(A - \lambda \cdot 1)$.

There is a nice generalization of this example to all semisimple Lie algebras due to Kostant (see [**Ko2**]). ♢

Example 4. Let N_+ (resp. N_-) be the subgroup of strictly upper (resp. lower) triangular matrices from $GL(n, \mathbb{R})$. The group $G = N_+ \times N_-$ acts on $X = \mathrm{Mat}_n(\mathbb{R})$:

$$g = (n_+, n_-): \ A \mapsto n_- \cdot A \cdot n_+^{-1}.$$

Take the subspace of diagonal matrices as S. Then almost all G-orbits intersect S in a single point (the Gauss Lemma in linear algebra). But in this case polynomial functions on S extend to rational invariant functions on X.

Namely, let $\Delta_k(A)$ denote the principal minor of order k for a matrix A. It is a G-invariant polynomial on X. Denote by f_k the function on S that is equal to the k-th diagonal element. Then the restriction of Δ_k to S is the product $f_1 f_2 \cdots f_k$. We see that the function f_k extends to X as a rational function Δ_k / Δ_{k-1}. ♢

In the case of the coadjoint action the polynomial and rational invariants play an important role in representation theory due to their connection with infinitesimal characters (see the next chapters). Here we remark only that smooth $K(G)$-invariants on $\mathfrak{g}^*$ form the center of the Lie algebra $C^\infty(\mathfrak{g}^*)$ with respect to Poisson brackets.

Indeed, this center consists of functions f such that

$$c_{ij}^k X_k \partial^i f = 0, \quad 1 \le j \le n.$$

But this means exactly that f is annihilated by all Lie vector fields $K_*(X_j)$, $1 \le j \le n$, hence is $K(G)$-invariant.

4. The moment map

4.1. The universal property of coadjoint orbits.

We have seen that any coadjoint orbit is a homogeneous symplectic manifold. The converse is "almost true": up to some algebraic and topological corrections (see below for details) any homogeneous symplectic manifold is a coadjoint orbit.

This theorem looks more natural in the context of Poisson G-manifolds (see Appendix II.3.2 for the introduction to Poisson manifolds).

In this section we always assume that G is connected.

Let us define a **Poisson G-manifold** as a pair $(M,\ f^M_{(\cdot)})$ where M is a Poisson manifold with an action of G and $f^M_{(\cdot)} : \mathfrak{g} \to C^\infty(M) : X \mapsto f^M_X$ is a homomorphism of Lie algebras such that the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "L_{(\cdot)}"] \arrow[dr, "f^M_{(\cdot)}"'] & Vect(M) \\ & C^\infty(M) \arrow[u, "\text{s-grad}"'] \end{tikzcd} \tag{19}$$

where L_X is the Lie field on M associated with $X \in \mathfrak{g}$ and s-grad(f) denotes the **skew gradient** of a function f, i.e. the vector field on M such that s-grad$(f)g = \{f, g\}$ for all $g \in C^\infty(M)$.

For a given Lie group G the collection of all Poisson G-manifolds forms the category $\mathcal{P}(G)$ where a morphism $\alpha : \left(M, f^M_{(\cdot)}\right) \to \left(N, f^N_{(\cdot)}\right)$ is a smooth map from M to N which preserves the Poisson brackets: $\{\alpha^*(\phi),\ \alpha^*(\psi)\} = \alpha^*(\{\phi, \psi\})$ and makes the following diagram commutative:

$$\begin{tikzcd} C^\infty(N) \arrow[r, "\alpha^*"] & C^\infty(M) \\ \mathfrak{g} \arrow[u, "f^N_{(\cdot)}"] \arrow[r, "id"] & \mathfrak{g} \arrow[u, "f^M_{(\cdot)}"'] \end{tikzcd} \tag{20}$$

Observe that the last condition implies that α commutes with the G-action.

An important example of a Poisson G-manifold is the space $(\mathfrak{g}^*, c)$ considered in Section 2.2 with the map $\mathfrak{g} \to C^\infty(\mathfrak{g}^*)$ defined by $f_X^{\mathfrak{g}^*}(F) = \langle F, X\rangle$.

Theorem 4. *The Poisson G-manifold $(\mathfrak{g}^*, c)$ is a universal (final) object in the category $\mathcal{P}(G)$.*

This means that for any object $(M, f_{(\cdot)}^M)$ there exists a unique morphism μ from $\big(M, f_{(\cdot)}^M\big)$ to $\big(\mathfrak{g}^*, f_{(\cdot)}^{\mathfrak{g}^*}\big)$, namely, the so-called **moment map** defined by

$$\langle \mu(m), X\rangle = f_X^M(m). \tag{21}$$

Proof. A direct corollary of the property (20) of a morphism in the category $\mathcal{P}(G)$.

Lemma 6. *Let M be a homogeneous Poisson G-manifold. Then the moment map is a covering of a coadjoint orbit.*

Proof. First, the image $\mu(M)$ is a homogeneous submanifold in $\mathfrak{g}^*$, i.e. a coadjoint orbit $\Omega \subset \mathfrak{g}^*$.

Second, the transitivity of the G-action on M implies that the rank of the Jacobi matrix for μ is equal to $\dim M$. Hence, μ is locally a diffeomorphism and globally a covering of Ω. □

It is worthwhile to discuss here the relation between homogeneous Poisson G-manifolds and homogeneous symplectic G-manifolds.

From Lemma 6 we see that all homogeneous Poisson G-manifolds are homogeneous symplectic G-manifolds. We show here that the converse is also true when coadjoint orbits are simply connected. In general, the converse statement becomes true after minor corrections.

This I call the **universal property** of coadjoint orbits.

Any symplectic manifold (M, σ) has a canonical Poisson structure c (Appendix II.3.2). But if a Lie group G acts on M and preserves σ, it does not imply that (M, c) is a Poisson G-manifold. Indeed, there are two obstacles for this:

1. *Topological obstacle.* The Lie field L_X, $X \in \mathfrak{g}$, is locally a skew gradient of some function f_X, but this function may not be defined globally.

To overcome this obstacle, we can consider an appropriate covering $\widetilde{M}$ of M where all f_X, $X \in \mathfrak{g}$, are single-valued. It may happen that the initial group G is not acting on $\widetilde{M}$ and must be replaced by some covering group $\widetilde{G}$.

The simply connected coverings of M and G are always sufficient.

2. *Algebraic obstacle.* The map $X \mapsto f_X$ of $\mathfrak{g}$ to $C^\infty(M)$ can always be chosen to be linear. Indeed, let $\{X_i\}$ be a basis in $\mathfrak{g}$. We choose functions f_i such that $L_{X_i} = \text{s-grad}\, f_i$ and put $f_X = \sum_i c_i f_i$ for $X = \sum_i c_i X_i$. But in general this is not a Lie algebra homomorphism: $f_{[X,Y]}$ and $\{f_X, f_Y\}$ can differ by a constant $c(X, Y)$.

Exercise 3. a) Show that the map $c : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} : (X, Y) \mapsto c(X, Y)$ is a 2-cocycle on $\mathfrak{g}$, i.e. satisfies the cocycle equation

$$\circlearrowleft c([X, Y], Z) = 0 \quad \text{for all } X, Y, Z \in \mathfrak{g}.$$

b) Check that the freedom in the choice of f_X is not essential for the cohomology class of c. In other words, the cocycle corresponding to a different choice of f_X differs from the original one by a trivial cocycle (or a coboundary of a 1-cycle b):

$$db(X, Y) = \langle b, [X, Y] \rangle$$

where $b \in \mathfrak{g}^*$ is a linear functional on $\mathfrak{g}$. ♣

To cope with the algebraic obstacle we have to pass from the initial Lie algebra $\mathfrak{g}$ to its central extension $\widetilde{\mathfrak{g}}$ given by the cocycle c. By definition, $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$ as a vector space. The commutator in $\widetilde{\mathfrak{g}}$ is defined by

$$[(X, a), (Y, b)] = \big([X, Y], c(X, Y)\big). \tag{22}$$

We define the action of $\widetilde{\mathfrak{g}}$ on M by $L_{(X,a)} := L_X$. So, practically it is the action of $\mathfrak{g} = \widetilde{\mathfrak{g}}/\mathbb{R}$, which is of no surprise because the center acts trivially in the adjoint and coadjoint representations.

We claim that this action is Poisson. Namely, we put

$$f_{(X,a)} := f_X + a$$

and a simple computation shows that $f_{[(X,a),(Y,b)]} = \{f_{(X,a)}, f_{(Y,b)}\}$.

The final conclusion is

Proposition 4. *Any symplectic action of a connected Lie group G on a symplectic manifold (M, σ) can be modified to a Poisson action of a central extension $\widetilde{G}$ of G on some covering $\widetilde{M}$ of M so that the following diagram is commutative:*

$$\begin{tikzcd} \widetilde{G} \times \widetilde{M} \arrow[r] \arrow[d] & \widetilde{M} \arrow[d] \\ G \times M \arrow[r] & M \end{tikzcd} \tag{23}$$

Here the horizontal arrows denote the actions and the vertical arrows denote the natural projections. □

4.2. Some particular cases.

For most "classical" (or "natural") groups the classification of coadjoint orbits is equivalent to one or another already known problem. In some cases, especially for infinite-dimensional groups, new interesting geometric and analytic problems arise. We discuss here only a few examples. Some others will appear later.

Example 5. Let $G = GL(n, \mathbb{R})$. This group is neither connected nor simply connected. So, we introduce $G^0 = GL_+(n, \mathbb{R})$, the connected component of unity in G, and denote by $\widetilde{G^0}$ the universal cover of G^0. It is worthwhile to note that G is homotopically equivalent to its maximal compact subgroup $O(n, \mathbb{R})$, while G^0 is equivalent to $SO(n, \mathbb{R})$ and $\widetilde{G^0}$ is equivalent to $\mathrm{Spin}(n, \mathbb{R})$ for $n \geq 3$. This follows from the well-known unique decomposition $g = kp$, $g \in G$, $k \in K$, $p \in P$, where $G = GL(n, \mathbb{R})$, $K = O(n, \mathbb{R})$ and P is the set of symmetric positive definite matrices.

The case $n = 2$ is a sort of exception. Here G^0 is diffeomorphic to $S^1 \times \mathbb{R}^3$ and $\widetilde{G^0}$ is diffeomorphic to $\mathbb{R}^4$.

As we mentioned in Section 1.1, the Lie algebra $\mathfrak{g} = \mathrm{Mat}_n(\mathbb{R})$ possesses an $\mathrm{Ad}(G)$-invariant bilinear form

$$\langle A, B\rangle = \mathrm{tr}(AB).$$

Thus, the coadjoint representation is equivalent to the adjoint one. Moreover, because the center acts trivially, the coadjoint action of $\widetilde{G^0}$ factors through G^0 and even through $G^0/\mathit{center} \simeq PSL(n, \mathbb{R})$. Therefore, coadjoint orbits for $\widetilde{G}^0$ are just G^0-conjugacy classes in $\mathrm{Mat}_n(\mathbb{R})$.

Since the Lie algebra $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) \simeq \mathbb{R} \oplus \mathfrak{sl}(n, \mathbb{R})$ has no non-trivial 1-cocycles, the algebraic obstacle is absent. So, all homogeneous symplectic G-manifolds are coverings of the G^0-conjugacy classes.

Exercise 4.* Show that every orbit is homotopic to one of the Stiefel manifolds $O(n_1 + \cdots + n_k)/\big(O(n_1) \times \cdots \times O(n_k)\big)$.

Hint. Use the information from Appendix I.2.3. ♣

Note that the fundamental group of an orbit is not necessarily commutative, e.g. for $n = 3$ there are orbits homotopic to

$$O(3)/\big(O(1) \times O(1) \times O(1)\big) \simeq U(1, \mathbb{H})/\{\pm 1, \pm i, \pm j, \pm k\}.$$

The fundamental group of these orbits is the so-called quaternionic group Q of order 8. ♢

Example 6. Let $G = SO(n, \mathbb{R})$. Here again, the group is not simply connected and we denote its universal cover by $\widetilde{G}_n$. It is known (and can be

derived from results in Appendix I.2.3) that

$$\pi_1\big(SO(n,\mathbb{R})\big) = \begin{cases} \mathbb{Z} & \text{for } n = 2, \\ \mathbb{Z}/2\mathbb{Z} & \text{for } n \geq 3. \end{cases}$$

For small values of n the group $\widetilde{G}_n$ is isomorphic to one of the following classical groups:

n	2	3	4	5	6
$\widetilde{G}_n$	$\mathbb{R}$	$SU(2)$	$SU(2) \times SU(2)$	$SU(2, \mathbb{H})$	$SU(4)$

For general $n \geq 3$ the group $\widetilde{G}_n$ is the so-called **spinor** group $\operatorname{Spin} n$. The most natural realization of this group can be obtained using the differential operators with polynomial coefficients on a supermanifold $\mathbb{R}^{0|n}$ (see, e.g., [**QFS**], vol. I).

The point is that the operations M_k of left multiplication by an odd coordinate ξ_k and the operations D_k of left differentiation with respect to ξ_k satisfy the **canonical anticommutation relations** (CAR in short):

$$M_i M_j + M_j M_i = D_i D_j + D_j D_i = 0; \qquad M_i D_j + D_j M_i = \delta_{i,j} \cdot 1.$$

The Lie algebra $\mathfrak{g} = \mathfrak{so}(n, \mathbb{R})$ is the set $\operatorname{Asym}_n(\mathbb{R})$ of all antisymmetric matrices X satisfying $X^t = -X$. The coadjoint action of $\widetilde{G}$ factors through G.

The restriction of the bilinear form above to $\mathfrak{g}$ is non-degenerate and $\operatorname{Ad}(G)$-invariant. So, the coadjoint representation is again equivalent to the adjoint one. The description of coadjoint orbits here is the problem of classification of antisymmetric matrices up to orthogonal conjugacy. In this case the orbits are simply connected and have the form

$$\Omega = SO(2n_1 + \cdots + 2n_k + m)/U(n_1) \times \cdots \times U(n_k) \times SO(m).$$

$\diamondsuit$

Example 7. Let $G = Sp(2n, \mathbb{R})$. It consists of matrices g satisfying

$$g^t J_n g = J_n \quad \text{where } J_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}.$$

The Lie algebra $\mathfrak{g}$ consists of J_n**-symmetric** matrices X satisfying

$$X^t J_n + J_n X = 0, \quad \text{or} \quad S = J_n X \quad \text{is symmetric}: \ S^t = S.$$

The classification of coadjoint orbits reduces in this case to the problem of classification of symmetric matrices up to transformations

$$S \mapsto g^t S g, \quad g \in Sp(2n, \mathbb{R}).$$

$\diamondsuit$

Remark 4. The problems arising in Examples 7 and 8 are particular cases of the following general problem: classification of a pair (S, A) where S is a symmetric matrix and A is an antisymmetric matrix with respect to simultaneous linear transformations:

$$(S, A) \mapsto (g^t S g, \ g^t A g), \quad g \in GL(n, \mathbb{R}).$$

$\heartsuit$

Example 8.* Let M be a compact smooth simply connected 3-dimensional manifold[5] with a given volume form vol. Let $G = \mathrm{Diff}(M, vol)$ be the group of volume preserving diffeomorphisms of M.

The role of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is played by the space $Vect(M, vol)$ of all divergence-free vector fields on M. We recall that the divergence of a vector field ξ with respect to a volume form vol is a function $div\ \xi$ on M such that $L_\xi(vol) = div\ \xi \cdot vol$. Here L_ξ is the Lie derivative along the field ξ. Using the identity (see formula (16) in Appendix II.2.3)

$$L_\xi = d \circ i_\xi \ + \ i_\xi \circ d$$

we obtain that $i_\xi vol = d\theta_\xi$ where θ_ξ is some 1-form on M defined modulo exact forms (differentials of functions). Now any smooth map $K : S^1 \to M$ defines a linear functional F_K on $\mathfrak{g}$:

$$F_K(\xi) = \int_{S^1} K^*(\theta_\xi). \tag{24}$$

(It is clear that adding a differential of a function to θ_ξ does not change the value of the integral.) Moreover, the functional F_K does not change if we reparametrize S^1 so that the orientation is preserved. In other words, it depends only on the oriented curve $K(S^1)$.

We see that the classification of coadjoint orbits in this particular case contains as a subproblem the classification of oriented knots in M up to a volume preserving isotopy. $\diamondsuit$

Example 9.* Let $G = \mathrm{Diff}_+(S^1)$ denote the group of orientation preserving diffeomorphisms of the circle, and let $\widetilde{G}$ be its simply connected

[5] The famous Poincaré conjecture claims that such a manifold is diffeomorphic to S^3 but it is still unknown. We use only the equalities $H^2(M) = H^1(M) = \{0\}$.

cover. This group has a unique non-trivial central extension G, the so-called **Virasoro-Bott group**.

This example will be discussed in Chapter 6. Here we show only that the classification of coadjoint orbits for this group is equivalent to each of the following apparently non-related problems.

1. Consider the ordinary differential equation of the second order

$$Ly \equiv cy'' + p(x)y = 0. \tag{25}$$

If we change the independent variable: $x \mapsto \phi(t)$, then equation (25) changes its form: the term with y' appears.

But if, at the same time, we change the unknown function: $y \mapsto y \circ \phi \cdot (\phi')^{-\frac{1}{2}}$, then the unwanted term with y' disappears and equation (25) goes to the equation $\tilde{L}\tilde{y} = 0$ of the same form but with a new coefficient

$$\tilde{p} = p \circ \phi \cdot (\phi')^2 + cS(\phi) \quad \text{where } S(\phi) = \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2. \tag{26}$$

Assume now that the coefficient $p(x)$ is 2π-periodic and the function $\phi(t)$ has the property $\phi(t + 2\pi) = \phi(t) + 2\pi$. The problem is to classify the equations (25) with respect to the transformations (26).

2. Let G be the simply connected covering of the group $SL(2, \mathbb{R})$, and let A be the group of all automorphisms of G. The problem is to classify elements of G up to the action of A.

3. The **locally projective structure** on the oriented circle S^1 is defined by a covering of S^1 by charts $\{U_\alpha\}_{\alpha \in A}$ with local parameter t_α on U_α such that the transition functions $\phi_{\alpha\beta}$ are fractional-linear and orientation preserving. (This means that $t_\alpha = \frac{at_\beta + b}{ct_\beta + d}$ with $ad - bc > 0$.)

The problem is to classify the locally projective structures on S^1 up to the action of $\mathrm{Diff}_+(S^1)$. $\diamondsuit$

Let us make a general observation about the relation between the coadjoint orbits of a group G and of its central extension $\widetilde{G}$ by a 1-dimensional subgroup A. This observation will explain the relation between the coadjoint orbits of the Virasoro-Bott group and problem 1 in Example 9.

Let $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ be the Lie algebras of G and $\widetilde{G}$. As a vector space, $\widetilde{\mathfrak{g}}$ can be identified with $\mathfrak{g} \oplus \mathbb{R}$ so that the commutator looks like

$$[(X, a), (Y, b)] = ([X, Y], c(X, Y)) \tag{27}$$

where $c(X, Y)$ is the cocycle defining the central extension. It is an antisymmetric bilinear map from $\mathfrak{g} \times \mathfrak{g}$ to $\mathbb{R}$ satisfying the **cocycle equation**:

$$\circlearrowleft c([X, Y], Z) = 0. \tag{28}$$

Here as usual the sign $\circlearrowleft$ denotes the sum over cyclic permutations of three variables.

We identify $\widetilde{\mathfrak{g}}^*$ with $\mathfrak{g}^* \oplus \mathbb{R}$ and denote its general element by (F, α). The coadjoint action of $\widetilde{G}$ reduces to an action of G because the central subgroup A acts trivially.

Lemma 7. *The coadjoint actions of G on $\widetilde{\mathfrak{g}}^*$ and on $\mathfrak{g}^*$ are related by the formula*

$$\widetilde{K}(g)\,(F, \alpha) = (K(g)F + \alpha \cdot S(g),\, \alpha) \tag{29}$$

where S is a 1-cocycle on the group G with values in $\mathfrak{g}^*$, i.e. a solution to the cocycle equation

$$S(g_1 g_2) = S(g_1) + K(g_1)S(g_2). \tag{30}$$

Proof. Since the extension $\widetilde{\mathfrak{g}}$ is central, the action of G on the quotient space $\widetilde{\mathfrak{g}}^*/\mathfrak{g}^*$ is trivial. Therefore, $\widetilde{K}(g)$ preserves hyperplanes $\alpha = const$ and on the hyperplane $\alpha = 0$ coincides with the ordinary coadjoint action of G on $\mathfrak{g}^*$. Hence, $\widetilde{K}(g)$ has the form (29) for some map $S : G \to \mathfrak{g}^*$. The cocycle property (30) follows directly from multiplicativity of the map $\widetilde{K}$.□

Exercise 5. Show that for any connected Lie group G the map S in (29) can be reconstructed from the cocycle $c(X, Y)$ entering in (27) as follows.

For any $g \in G$ the cocycles $c(X, Y)$ and $c'(X, Y) = c(\operatorname{Ad} g\, X, \operatorname{Ad} g\, Y)$ are equivalent.[6] Thus, we can write

$$c(\operatorname{Ad} g\, X, \operatorname{Ad} g\, Y) = c(X, Y) + \langle \Phi(g), [X, Y] \rangle. \tag{31}$$

From this we derive that

$$\widetilde{\operatorname{Ad}}\, g(X, a) = (\operatorname{Ad} g\, X,\, a + \langle \Phi(g), X \rangle)$$

and, consequently, (29) follows with $S(g) = \Phi(g^{-1})$.

5. Polarizations

5.1. Elements of symplectic geometry.

We shall use here the general facts about symplectic manifolds from Appendix II.3: the notions of skew gradient, Poisson brackets, etc.

In the general scheme of geometric quantization (which is a quantum mechanical counterpart of the construction of unirreps from coadjoint orbits) the notion of a polarization plays an important role.

[6]The infinitesimal version of this statement follows directly from (28).

Definition 3. Let (M, σ) be a symplectic manifold. A **real polarization** of (M, σ) is an integrable subbundle P of the tangent bundle TM such that each fiber $P(m)$ is a maximal isotropic subspace in the symplectic vector space $(T_mM, \sigma(m))$. In particular, the dimension of P is equal to $\frac{1}{2}\dim M$.

Recall that a subbundle P is called **integrable** if there exists a foliation of M, i.e. a decomposition of M into disjoint parts, the so-called **leaves**, such that the tangent space to a leaf at any point $m \in M$ is exactly $P(m)$.

To formulate the necessary and sufficient conditions for the integrability of P we need some notation.

Let us call a vector field ξ on M **P-admissible** if $\xi(m) \in P(m)$ for all $m \in M$. The space of all P-admissible vector fields is denoted by $Vect_P(M)$.

The dual object is the ideal $\Omega_P(M)$ of all P-admissible differential forms ω on M which have the property:

$$\omega(\xi_1, \dots, \xi_k) = 0 \quad \text{for any } P\text{-admissible vector fields } \xi_1, \dots, \xi_k,\ k = \deg \omega.$$

Frobenius Integrability Criterion. *The following are equivalent:*

a) *A subbundle $P \subset TM$ is integrable.*

b) *The vector space $Vect_P(M)$ is a Lie subalgebra in $Vect(M)$.*

c) *The vector space $\Omega_P(M)$ is a differential ideal in the algebra $\Omega(M)$.*$\square$

In practice only those polarizations that are actually fibrations of M are used. In this case the set of leaves is itself a smooth manifold B and M is a fibered space over B with leaves as fibers. These leaves are **Lagrangian** (i.e. maximal isotropic) submanifolds of M.

Let $C_P^\infty(M)$ denote the space of smooth functions on M which are constant along the leaves. In fact it is a subalgebra in $C^\infty(M)$ which can also be defined as the set of functions annihilated by all admissible vector fields.

Lemma 8. *A subbundle $P \subset TM$ of dimension $\frac{1}{2}\dim M$ is a polarization iff $C_P^\infty(M)$ is a maximal abelian subalgebra in the Lie algebra $C^\infty(M)$ with respect to Poisson brackets.*

Proof. Assume that P is a polarization. The space $Vect_P(M)$ consists of vector fields tangent to the fibers of P. Therefore, for $f \in C_P^\infty(M)$ we have $df \in \Omega_P^1(M)$. It follows that s-grad $f(m)$ is σ-orthogonal to $P(M)$, hence belongs to $P(M)$. So, for any $f_1, f_2 \in C_P^\infty(M)$ we have $\{f_1, f_2\} = (\text{s-grad}\, f_1)f_2 = 0$.

Moreover, if $(\text{s-grad}\, f_1)f_2 = 0$ for all $f_2 \in C_P^\infty(M)$, then s-grad $f(m) \in P(m)$ and f_1 is constant along the fibers, hence belongs to $C_P^\infty(M)$. We have shown that $C_P^\infty(M)$ is a maximal abelian Lie subalgebra in $C^\infty(M)$.

Assume now that $C^\infty_P(M)$ is an abelian Lie subalgebra in $C^\infty(M)$. According to formula (32) from Appendix II.3.1 we have $\{f_1, f_2\} = \sigma(\text{s-grad} f_1, \text{s-grad} f_2)$. We see that skew-gradients of $f \in C^\infty_P(M)$ span an isotropic subspace at every point $m \in M$. But this subspace has dimension $\frac{1}{2}\dim M$, hence must be a maximal isotropic subspace in $T_m(M)$. $\square$

There is a remarkable complex analog of real polarizations.

Definition 4. A **complex polarization** of (M, σ) is an integrable subbundle P of the complexified tangent bundle $T^{\mathbb{C}}M$ such that each fiber $P(m)$ is a maximal isotropic subspace in the symplectic complex vector space $(T^{\mathbb{C}}_m M, \sigma^{\mathbb{C}}(m))$.

Here the integrability is defined formally by the equivalent conditions b) and c) in the Frobenius Criterion above.

The space $C^\infty_P(M)$, as before, is a subalgebra in the complexification of $C^\infty(M)$. A simple description of this subalgebra can be given in a special case.

Let P be an integrable complex subbundle of $T^{\mathbb{C}}M$. Then its complex conjugate $\overline{P}$ and the intersection $D := P \cap \overline{P}$ are also integrable (this is an easy exercise in application of the Frobenius Criterion). On the contrary, the subbundle $E := P + \overline{P}$ in general is not integrable.

Note that both D and E are invariant under complex conjugation, hence can be viewed as complexifications of real subbundles $D_0 = D \cap TM$ and $E_0 = E \cap TM$, respectively.

Proposition 5. *Assume that the subbundle E_0 is integrable. Then in a neighborhood of every point of M there exists a local coordinate system $\{u_1, \dots, u_k;\ x_1, \dots, x_l;\ y_1, \dots, y_l;\ v_1, \dots, v_m\}$ with the following properties:*

(i) *D_0 is generated by $\frac{\partial}{\partial v_i}$, $1 \le i \le k$;*

(ii) *E_0 is generated by $\frac{\partial}{\partial v_i}$, $1 \le i \le k$, $\frac{\partial}{\partial x_j}$, $1 \le j \le l$, and $\frac{\partial}{\partial y_j}$, $1 \le j \le l$;*

(iii) *P is generated by $\frac{\partial}{\partial v_i}$, $1 \le i \le k$, and $\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}$, $1 \le j \le l$.* $\square$

The crucial case is $D_0 = 0$, $E_0 = TM$. In this case Proposition 5 is exactly the Nirenberg-Newlander theorem on integrability of an almost complex structure.

Let us introduce the notation $z_j = x_j + iy_j$, $1 \le j \le l$. Then we can say that the algebra $C^\infty_P(M)$ consists of functions which do not depend on coordinates v_i, $1 \le i \le k$, and are holomorphic in coordinates z_j, $1 \le j \le l$.

Or, more geometrically, the functions in question are constant along leaves of D_0 and holomorphic along leaves of E_0/D_0.

Remark 5. The subbundles $P \subset T^{\mathbb{C}}M$ for which $E = P + \overline{P}$ is not integrable are rather interesting, but until now have not been used in representation theory. In this case $C^{\infty}_{P}(M)$ is still a subalgebra in $C^{\infty}(M)$. The nature of this subalgebra can be illustrated in the following simple example.[7]

Let $M = \mathbb{R}^3$ with coordinates x, y, t, and let P be spanned by a single complex vector field $\xi = \partial_x + i\partial_y + (y - ix)\partial_t$. In terms of the complex coordinate $z = x + iy$ it can be rewritten as $\xi = \partial_{\overline{z}} - iz\partial_t$. Then $\overline{P}$ is spanned by $\overline{\xi} = \partial_z + i\overline{z}\partial_t$, and $E = \mathbb{C} \cdot \xi \oplus \mathbb{C} \cdot \overline{\xi}$ is not integrable since $[\xi, \overline{\xi}] = 2i\partial_t \notin E$.

The equation $\xi f = 0$ is well known and rather famous in analysis. The point is that the corresponding non-homogeneous equation $\xi f = g$ has no solution for most functions g.

Some of the solutions to the equation $\xi f = 0$ have a transparent interpretation. Consider the domain

$$D = \{(z, w) \in \mathbb{C}^2 \mid \operatorname{Im} w \geq |z|^2\}.$$

The boundary ∂D is diffeomorphic to $\mathbb{R}^3$ and is naturally parametrized by coordinates z and $t = \operatorname{Re} w$. It turns out that the boundary values of holomorphic functions in D satisfy the equation $\xi f = 0$. However, they do not exhaust all the solutions which can be non-analytic in a real sense. ♡

5.2. Invariant polarizations on homogeneous symplectic manifolds.

In the representation theory of Lie groups one is interested mainly in G-invariant polarizations of homogeneous symplectic G-manifolds.

We know also that the latter are essentially coadjoint orbits. In this situation the geometric and analytic problems can be reduced to pure algebraic ones. Let G be a connected Lie group, and let $\Omega \subset \mathfrak{g}^*$ be a coadjoint orbit of G. Choose a point $F \in \Omega$ and denote by $Stab(F)$ the stabilizer of F in G and by $stab(F)$ its Lie algebra.

Definition 5. We say that a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is **subordinate** to a functional $F \in \mathfrak{g}^*$ if the following equivalent conditions are satisfied:

(i) $F \mid_{[\mathfrak{h},\mathfrak{h}]} = 0$;

(i′) the map $X \mapsto \langle F, X\rangle$ is a 1-dimensional representation of $\mathfrak{h}$.

Note that the codimension of $\mathfrak{h}$ in $\mathfrak{g}$ is at least $\frac{1}{2}\operatorname{rk} B_F$.

[7] In this example M is not a symplectic manifold but a so-called contact manifold. In a sense, contact manifolds are odd-dimensional analogues of symplectic manifolds.

We say that $\mathfrak{h}$ is a **real algebraic polarization** of F if in addition the condition

(ii) $\operatorname{codim}_{\mathfrak{g}} \mathfrak{h} = \frac{1}{2}\operatorname{rk} B_F$ (i.e. $\mathfrak{h}$ has maximal possible dimension $\frac{\dim \mathfrak{g} + \operatorname{rk} \mathfrak{g}}{2}$) is satisfied.

The notion of a **complex algebraic polarization** is defined in the same way: we extend F to $\mathfrak{g}_{\mathbb{C}}$ by complex linearity and consider complex subalgebras $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}}$ that satisfy the equivalent conditions (i) or (i′) and the condition (ii).

An algebraic polarization $\mathfrak{h}$ is called **admissible** if it is invariant under the adjoint action of $Stab(F)$. Note, that any polarization contains the Lie algebra $stab(F)$, hence is invariant under the adjoint action of $Stab^0(F)$, the connected component of unity in $Stab(F)$.

The relation of these "algebraic" polarizations to "geometric" ones defined earlier is very simple and will be explained later (see Theorem 5). It can happen that there is no real G-invariant polarization for a given $F \in \mathfrak{g}^*$. The most visual example is the case $G = SU(2)$ where $\mathfrak{g}$ has no subalgebras of dimension 2.

However, real G-invariant polarizations always exist for nilpotent and completely solvable Lie algebras while complex polarizations always exist for solvable Lie algebras. It follows from a remarkable observation by Michele Vergne.

Lemma 9 (see [Ver1, Di2]). *Let V be a real vector space endowed with a symplectic bilinear form B. Consider any filtration of V:*

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

where $\dim V_k = k$. Denote by W_k the kernel of the restriction $B\big|_{V_k}$. Then

a) *The subspace $W = \sum_k W_k$ is maximal isotropic for B.*

b) *If in addition V is a Lie algebra, $B = B_F$ for some $F \in V^*$ and all V_k are ideals in V, then W is a polarization for F.* □

Note that in [**Di2**] it is also shown that for a Lie algebra $\mathfrak{g}$ over an algebraically closed field K the set of functionals $F \in \mathfrak{g}^*$ that admit a polarization over K contains a Zariski open subset, hence is dense in $\mathfrak{g}^*$.

Example 10. Let $G = Sp(2n, K)$, $K = \mathbb{R}$ or $\mathbb{C}$. The Lie algebra $\mathfrak{g}$ consists of matrices of the form SJ where S is a symmetric matrix with elements from K and $J = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. The dual space $\mathfrak{g}^*$ can be identified with $\mathfrak{g}$ using the pairing $(X, Y) = \operatorname{tr}(XY)$. Consider the subset $\Omega \subset \mathfrak{g}^*$ given by the condition $\operatorname{rk} X = 1$. This set is a single G-orbit in the case $K = \mathbb{C}$ and splits into two G-orbits $\Omega_{\pm}$ in the case $K = \mathbb{R}$.

We show that for $n \geq 2$ there is no algebraic polarization for any $F \in \Omega$. Indeed, it is easy to compute that $\operatorname{rk} B_F = 2n$ for $F \in \Omega$. So, the would-be polarization $\mathfrak{h}$ must have codimension n in $\mathfrak{g}$. It also must contain $stab(F)$, hence, $\mathfrak{h}/stab(F)$ has to be an n-dimensional $Stab(F)$-invariant subspace in $T_F\Omega \cong \mathfrak{g}/stab(F) \cong K^{2n}$.

Consider the action of $Stab(F)$ on $T_F\Omega \cong K^{2n}$ in more detail. We can write the matrix F in the form $F = vv^tJ$ for some column vector $v \in K^{2n}$. Two vectors v and v' define the same functional F if $v = \pm v'$. Therefore $Stab(F)$ consists of matrices g satisfying $gv = \pm v$. The connected component of the unit in $Stab(F)$ is defined by the condition $gv = v$. It is the so-called **odd symplectic group** $Sp(2n-1, \mathbb{R})$ (see Section 6.2.2).

The map $v \mapsto F = vv^tJ$ is G-equivariant and identifies the linear action of $Stab(F)$ on $T_F\Omega$ with the standard one on K^{2n}. According to the Witt theorem,[8] this action admits only two non-trivial invariant subspaces: the 1-dimensional space Kv and the $(2n-1)$-dimensional space $(Kv)^\perp$. Hence, an n-dimensional invariant subspace can exist only for $n = 1$. ♢

Now we explain the relation between the notions of geometric and algebraic polarization. As before, denote by p_F the map from G onto Ω defined by $p_F(g) = K(g)F$, and by $(p_F)_*$ its derivative at e which maps $\mathfrak{g}$ onto $T_F\Omega$.

Theorem 5. *There is a bijection between the set of G-invariant real polarizations P of a coadjoint orbit $\Omega \subset \mathfrak{g}^*$ and the set of admissible real algebraic polarizations $\mathfrak{h}$ of a given element $F \in \Omega$.*

Namely, to a polarization $P \subset T\Omega$ there corresponds the algebraic polarization $\mathfrak{h} = (p_F)_*^{-1}(P(F))$.

Proof. We use the following general result

Proposition 6. *Let $M = G/K$ be a homogeneous manifold.*

a) *There is a one-to-one correspondence between G-invariant subbundles $P \subset TM$ and K-invariant subspaces $\mathfrak{h} \subset \mathfrak{g}$ containing* Lie(K).

b) *The subbundle P is integrable if and only if the corresponding subspace $\mathfrak{h}$ is a subalgebra in $\mathfrak{g}$.*

Proof. a) Choose an initial point $m_0 \in M$ and denote by p the projection of G onto M acting by the formula $p(g) = g \cdot m_0$. Let $p_*(g)$ be the derivative of p at the point $g \in G$. Define a subbundle $Q \subset TG$ by

$$Q(g) := p_*^{-1}(g)(P(g \cdot m_0)).$$

[8]The **Witt theorem** claims that in a vector space V with a symmetric or antisymmetric bilinear form B any partial B-isometry $A_0 : V_0 \to V$ can be extended to a global B-isometry $A : V \to V$.

The following diagram is clearly commutative:

$$
(32)\qquad
\begin{array}{ccccccc}
Q(e)=\mathfrak{h} & \longrightarrow & \mathfrak{g} & \xrightarrow{(L_g)_*(e)} & T_g(G) & \longleftarrow & Q(g) \\
{\scriptstyle p_*(e)}\big\downarrow & & {\scriptstyle p_*(e)}\big\downarrow & & \big\downarrow{\scriptstyle p_*(g)} & & \big\downarrow{\scriptstyle p_*(g)} \\
P(m_0) & \longrightarrow & T_{m_0}M & \xrightarrow{(g\,\cdot)_*(m_0)} & T_{g\cdot m_0}M & \longleftarrow & P(g\cdot m_0)
\end{array}
$$

(Here L_g is a left shift by $g \in G$ and an asterisk as a lower index means the derivative map.)

It follows that Q is a left-invariant subbundle of TG. Conversely, every left-invariant subbundle $Q \subset TG$ with the property $\mathfrak{h} := Q(e) \supset \mathrm{Lie}(K)$ can be obtained by this procedure from a G-invariant subbundle $P \subset TM$: we just define $P(g \cdot m_0)$ as $p_*(g)Q(g)$.

b) Being G-invariant, P is spanned by G-invariant vector fields $\widetilde{X}$, $X \in \mathfrak{h}$. By the Frobenius criterion P is integrable iff the space $Vect_P(M)$ is a Lie subalgebra. But the last condition is equivalent to the claim that $\mathfrak{h}$ is a Lie subalgebra in $\mathfrak{g}$. □

Let us return to the proof of Theorem 5.

For a given subbundle $P \subset T\Omega$ we define $\mathfrak{h}$ as in Proposition 6 (with Ω in the role of M and F in the role of m_0). We saw in Section 2.1 that $p^*(e)\sigma(F) = B_F$. Therefore $P(F)$ is maximal isotropic with respect to σ iff the same is true for $\mathfrak{h}$ with respect to B_F. The remaining statements of Theorem 5 follow from Proposition 6. □

Remark 6. Let $\mathfrak{h}$ be a real polarization of $F \in \mathfrak{g}^*$, let P be the real polarization of Ω which corresponds to $\mathfrak{h}$, and let $H = \exp \mathfrak{h}$. Then the leaves of the G-invariant foliation of Ω determined by P have the form $K(gH)(F)$. In particular, the leaf passing through F is an orbit of coadjoint action of H. ♡

In conclusion we note that Theorem 5 can be easily reformulated and proved in the complex case. It looks as follows.

Theorem 5′. *There is a bijection between the set of all G-invariant complex polarizations P of a coadjoint orbit $\Omega \subset \mathfrak{g}^*$ and the set of all complex algebraic polarizations $\mathfrak{h}$ of a given element $F \in \Omega$. As before, to a polarization $P \subset T_\Omega^{\mathbb{C}}$ there corresponds the subalgebra $\mathfrak{h} = p_*(F)^{-1}(P(F)) \subset \mathfrak{g}^{\mathbb{C}}$.* □

Chapter 2

Representations and Orbits of the Heisenberg Group

There are several reasons to consider the Heisenberg group separately and in detail before the exposition of the general theory.

First, it is the simplest non-abelian nilpotent Lie group, actually the only one of dimension 3. Therefore, it has many different realizations.

Second, it appears naturally in quantum mechanics. The description of all unitary representations of the Heisenberg group is essentially equivalent to the description of all realizations of the canonical commutation relations.

Third, representation theory for the Heisenberg group enters as a "building block" in the general theory, like representations of $SL(2)$ are widely used in the general theory of semisimple groups.

Finally, the Heisenberg Lie algebra is a contraction of many other important Lie algebras. Therefore, the formulae concerning unitary representations of the Heisenberg group can be considered as limit cases of the corresponding formulae for $SL(2, \mathbb{R})$, $SU(2, \mathbb{C})$, and groups of motions of the Euclidean and pseudo-Euclidean plane.

1. Heisenberg Lie algebra and Heisenberg Lie group

1.1. Some realizations.

The **Heisenberg Lie algebra** $\mathfrak{h}$ is a 3-dimensional real vector space with basic vectors X, Y, Z, satisfying the following commutation relations:

$$[X, Y] = Z, \qquad [X, Z] = [Y, Z] = 0. \tag{1}$$

It has a simple matrix realization by upper triangular 3×3 matrices:

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

or

$$xX + yY + zZ = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

Sometimes it is more convenient to consider a 4-dimensional realization:

$$xX + yY + zZ = \begin{pmatrix} 0 & x & y & 2z \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

In particular, this realization suggests the natural generalization. Let $\mathfrak{h}_n$ denote the $(2n+1)$-dimensional algebra of block-triangular matrices of the form

$$\begin{pmatrix} 0 & \vec{x}^{\,t} & \vec{y}^{\,t} & 2z \\ 0 & 0 & 0 & \vec{y} \\ 0 & 0 & 0 & -\vec{x} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

where $\vec{x}$, $\vec{y}$ are column vectors and $\vec{x}^{\,t} = (x_1, \ldots, x_n)$ and $\vec{y}^{\,t} = (y_1, \ldots, y_n)$ are the transposed row vectors.

We call $\mathfrak{h}_n$ a **generalized Heisenberg Lie algebra**. From the point of view of the general theory of Lie groups, $\mathfrak{h}_n$ is the nilpotent radical of a maximal parabolic subalgebra $\mathfrak{p}_n$ in the symplectic Lie algebra $\mathfrak{sp}(2n+2, \mathbb{R})$. It is also a non-trivial central extension of the abelian Lie algebra $\mathbb{R}^{2n}$ by a 1-dimensional center.

It has the natural basis $\{X_i,\ Y_i,\ 1 \le i \le n,\ Z\}$ with the commutation relations:

$$\begin{aligned} &[X_i, X_j] = [Y_i, Y_j] = [X_i, Z] = [Y_i, Z] = 0, \\ &[X_i, Y_j] = \delta_{ij} Z. \end{aligned} \tag{5}$$

Practically, all we can say about the Heisenberg Lie algebra $\mathfrak{h}$ can be extended to the general $\mathfrak{h}_n$. Moreover, in the appropriate notation the corresponding formulae look exactly the same as (3) and (4) above.

Let H_n be the simply connected Lie group with $\mathrm{Lie}\,(H_n) = \mathfrak{h}_n$. We call it a **generalized Heisenberg group**. For $n = 1$ we call it simply the **Heisenberg group** and denote it by H.

The group H_n can be realized by matrices

$$(6)\quad h(\vec{x}, \vec{y}, z) = \exp\left(\sum_{i=1}^{n}(x_iX_i + y_iY_i) + zZ\right) = \begin{pmatrix} 1 & \vec{x}^{\,t} & \vec{y}^{\,t} & 2z \\ 0 & 1 & 0 & \vec{y} \\ 0 & 0 & 1 & -\vec{x} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that the exponential map defines a global coordinate system on H_n, the so-called **exponential coordinates**. Hence, H_n is an exponential Lie group.

Sometimes other coordinate systems are useful, e.g. one can write an element $h \in H$ as

$$\widetilde{h}(a, b, c) = \exp aX \exp bY \exp cZ, \qquad a, b, c \in \mathbb{R}.$$

The triple (a, b, c) form **canonical coordinates** relative to the ordered basis $\{X, Y, Z\}$ in $\mathfrak{h}$.

Exercise 1. Find the relation between exponential coordinates (x, y, z) and canonical coordinates (a, b, c) in H.

Answer: $a = x, \quad b = y, \quad c = z - \frac{1}{2}xy.$ ♣

Exercise 2. Describe all closed connected subgroups in H and determine which of them are normal.

Hint. All such subgroups have the form $A = \exp \mathfrak{a}$ where $\mathfrak{a} \subset \mathfrak{h}$ is a Lie subalgebra. Normal subgroups in H correspond to ideals in $\mathfrak{h}$.

Answer: The list of subgroups:

a) The trivial subgroup $\{e\}$.

b) One-dimensional subgroups $\{\exp\ L\}$ where L is a line (i.e. one-dimensional subspace) in $\mathfrak{h}$.

c) Two-dimensional subgroups $\exp P$ where P is any plane (i.e. two-dimensional subspace in $\mathfrak{h}$) that contains the line $\mathbb{R} \cdot Z$.

d) The whole group H.

The list of normal subgroups:

All subgroups of dimension $0, 2$, or 3 and the 1-dimensional subgroup $C = \exp \mathbb{R} \cdot Z$, which is the center of H. ♣

Exercise 3. Consider the space $\mathbb{R}^{2n}$ endowed with the standard symplectic form $\sigma = \sum_{i=1}^{n} (dx_i \wedge dy_i)$. Show that polynomials of degree ≤ 1 form a Lie algebra with respect to Poisson brackets (see Appendix II.3.2 for the definition of Poisson brackets). Check that this Lie algebra is isomorphic to $\mathfrak{h}_n$. ♣

Exercise 4. Consider the set Φ of all smooth maps $x \mapsto \phi(x)$ defined in some neighborhood of zero and subjected to the conditions $\phi(0) = 0$, $\phi'(0) = 1$. Fix some natural number N and say that two maps ϕ_1 and ϕ_2 are equivalent if $\phi_1^{(k)}(0) = \phi_2^{(k)}(0)$ for $k = 2, 3, \ldots, N+1$.

Let $G_N = \Phi/\!\sim$ be the set of equivalence classes. It is convenient to write an equivalence class $g \in G_N$ as a formal transformation

$$g(x) = x + a_1 x^2 + a_2 x^3 + \cdots + a_N x^{N+1} + o(x^{N+1}).$$

a) Show that G_N is an N-dimensional Lie group with global coordinate system $(a_1, \ldots, a_N)$ where the multiplication law is defined by composition of transformations.

b) Let $\mathfrak{g}_N = \mathrm{Lie}(G_N)$. Find the commutation relations in an appropriate basis in $\mathfrak{g}_N$.

c) Show that the group G_3 is isomorphic to the Heisenberg group H.

Hint. Use the realization of $\mathrm{Lie}(G_3)$ by equivalence classes of formal vector fields

$$v = \left(\alpha x^2 + \beta x^3 + \gamma x^4 + o(x^4)\right) \frac{d}{dx}$$

and prove the relations

$$\left[x^k \frac{d}{dx},\, x^l \frac{d}{dx}\right] = (l-k) x^{k+l-1} \frac{d}{dx}.$$

♣

Exercise 5.* Show that for any real $x \in (0, \pi)$

$$\underbrace{\sin\,\sin \cdots \sin}_{N \text{ times}}(x) \approx \frac{c}{N^\alpha}$$

for some positive constants c and α. Find these constants.

Hint. Use the fact that the flow corresponding to the vector field $v = x^3 \frac{d}{dx}$ has the explicit form

$$\Phi_t(x) = \frac{x}{\sqrt{1 - 2tx^2}} = x + tx^3 + o(|t|).$$

It follows that for any $\epsilon > 0$ the inequalities

$$\Phi_{-\frac{1}{6}-\epsilon}(x) \le \sin x \le \Phi_{-\frac{1}{6}+\epsilon}(x)$$

hold for $x > 0$ in some neighborhood of 0. Derive from it the estimation

$$\Phi_{-\frac{N}{6}-\epsilon}(x) \le \underbrace{\sin \sin \cdots \sin}_{N \text{ times}}(x) \le \Phi_{-\frac{N}{6}+\epsilon}(x).$$

Answer: $\alpha = \frac{1}{2}$, $c = \sqrt{3}$. In particular, you can check on your calculator that $\underbrace{\sin \sin \cdots \sin}_{100 \text{ times}}(x) \approx 0.17$ for a randomly chosen $x \in (0, 1)$. ♣

1.2. Universal enveloping algebra $U(\mathfrak{h})$.

The universal enveloping algebra $U(\mathfrak{h})$ is by definition (see Appendix III.1.4) an associative algebra with unit over $\mathbb{C}$ which has the same generators X, Y, Z as the Lie algebra $\mathfrak{h}$, subjected to the relations

$$(7) \qquad XY - YX = Z, \qquad XZ = ZX, \qquad YZ = ZY.$$

So, the elements of $U(\mathfrak{h})$ are "non-commutative polynomials" in X, Y, Z.

There are three natural bases in the vector space $U(\mathfrak{h})$. All three establish a vector space isomorphism between $U(\mathfrak{h})$ and the ordinary polynomial algebra $\mathbb{C}[x, y, z] \simeq S(\mathfrak{h})$.

First, according to the Poincaré-Birkhoff-Witt theorem, for any basis A, B, $C \in \mathfrak{h}$ the monomials $A^k B^l C^m$, $k, l, m \in \mathbb{Z}_+$, form a basis in $U(\mathfrak{h})$. In particular, we get two isomorphisms

$$\begin{aligned} a:\ \mathbb{C}[x, y, z] \to U(\mathfrak{h}):&\quad x^k y^l z^m \mapsto X^k Y^l Z^m, \\ b:\ \mathbb{C}[x, y, z] \to U(\mathfrak{h}):&\quad x^k y^l z^m \mapsto Y^l X^k Z^m. \end{aligned}$$

Moreover, we also have the **symmetrization map**

(8)

$$\mathbf{sym}:\ \mathbb{C}[x, y, z] \to U(\mathfrak{h})$$

$$x^k y^l z^m \mapsto \text{the coefficient at } \frac{\alpha^k \beta^l \gamma^m}{k!\, l!\, m!} \text{ in } \frac{(\alpha X + \beta Y + \gamma Z)^{k+l+m}}{(k+l+m)!}.$$

An equivalent formulation is

$$(8') \qquad \mathbf{sym}\big(P(x, y, z)\big) = P\left(\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}, \frac{\partial}{\partial\gamma}\right) e^{\alpha X + \beta Y + \gamma Z}\Big|_{\alpha=\beta=\gamma=0}.$$

Exercise 6. Show that the map **sym** has the property

$$\mathbf{sym}(a_1 a_2 \cdots a_n) = \frac{1}{n!} \sum_{s \in S_n} \mathbf{sym}\big(a_{s(1)}\big)\mathbf{sym}\big(a_{s(2)}\big) \cdots \mathbf{sym}\big(a_{s(n)}\big) \tag{9}$$

where $a_1, \ldots, a_n$ are arbitrary linear combinations of x, y, z. In particular, we have

$$\mathbf{sym}\big((ax + by + cz)^n\big) = (aX + bY + cZ)^n. \tag{9'}$$

Hint. Use the fact that both sides of (9) are polylinear in $a_1, \ldots, a_n$. Therefore, it suffices to check it for the case when all the a_i take values x, y, z. ♣

There is a beautiful relation between these three maps. Let D be a linear operator in $\mathbb{C}[x,\, y,\, z]$ given by the differential operator of infinite order

$$D = e^{\frac{z}{2} \cdot \frac{\partial^2}{\partial x\, \partial y}} = 1 + \frac{z}{2} \cdot \frac{\partial^2}{\partial x\, \partial y} + \frac{z^2}{8} \cdot \frac{\partial^4}{(\partial x)^2\, (\partial y)^2} + \cdots .$$

Proposition 1. *The following diagram is commutative:*

$$\begin{array}{ccccc} \mathbb{C}[x,\, y,\, z] & \xrightarrow{D} & \mathbb{C}[x,\, y,\, z] & \xrightarrow{D} & \mathbb{C}[x,\, y,\, z] \\ {\scriptstyle a}\big\downarrow & & {\scriptstyle \mathbf{sym}}\big\downarrow & & \big\downarrow{\scriptstyle b} \\ U(\mathfrak{h}) & \xrightarrow[\mathrm{Id}]{} & U(\mathfrak{h}) & \xrightarrow[\mathrm{Id}]{} & U(\mathfrak{h}) \end{array} \tag{10}$$

Proof. One of the simplest proofs can be obtained using the realization of $U(\mathfrak{h})$ by differential operators acting on $\mathbb{C}[s,\, t]$:

$$X \mapsto \frac{d}{dt}, \qquad Y \mapsto st, \qquad Z \mapsto s.$$

Then

$$a(x^k y^l z^m) = \left(\frac{d}{dt}\right)^k \circ (st)^l \circ s^m = \sum_{i \geq 0} \binom{k}{i} \frac{l!}{(l-i)!} t^{l-i} \left(\frac{d}{dt}\right)^{k-i} s^{m+l},$$

$$\begin{aligned} \left((b \circ D^2)\right)\left(x^k y^l z^m\right) &= \sum_{i \geq 0} \frac{k!\, l!}{i!\, (k-i)!\, (l-i)!} b\left(x^{k-i} y^{l-i} z^{m+i}\right) \\ &= \sum_{i \geq 0} \frac{k!\, l!}{i!\, (k-i)!\, (l-i)!} (st)^{l-i} \left(\frac{d}{dt}\right)^{k-i} s^{m+i}, \end{aligned}$$

and we see that $a = b \circ D^2$. The equalities $a = \mathbf{sym} \circ D$ and $\mathbf{sym} = b \circ D$ can be proved in the same way. □

We see that **sym** is sort of a "geometric mean" between a and b such that we have the proportion

$$a : \mathbf{sym} \;=\; \mathbf{sym} : b \;=\; D^{-1}.$$

A useful consequence of Proposition 1 is the following formula:

$$X{\cdot}\mathbf{sym}(P) = \mathbf{sym}(DxD^{-1}P) = \mathbf{sym}\left(\left(x + \frac{1}{2}z\partial_y\right)P\right) \quad \text{for } P \in \mathbb{C}[x,y,z],$$

which follows from the obvious relation $a(xP) = Xa(P)$.

Using this formula, we can reduce to the symmetric form the product of two elements of $U(\mathfrak{h})$ written in the symmetric form. The result is

$$\mathbf{sym}^{-1}\big(\mathbf{sym}(P)\cdot\mathbf{sym}(Q)\big) = \sum_{k=0}^{\infty}\frac{1}{k!}P \overset{k}{\circ} Q \tag{11}$$

where the k-th multiplication law $\overset{k}{\circ}$ is defined by

$$P \overset{k}{\circ} Q = \left(\frac{z}{2}\right)^k \cdot \sum_{a=0}^{k}\binom{k}{a}(-1)^a \frac{\partial^k P}{(\partial x)^{k-a}(\partial y)^a}\frac{\partial^k Q}{(\partial x)^a(\partial y)^{k-a}}. \tag{12}$$

In particular,

$$P \overset{0}{\circ} Q = PQ, \qquad P \overset{1}{\circ} Q = \frac{z}{2}\cdot\left(\frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Q}{\partial x}\right).$$

1.3. The Heisenberg Lie algebra as a contraction.

There is one more remarkable feature of the Heisenberg Lie algebra $\mathfrak{h}$: it appears as a contraction of several other Lie algebras.

The notions of deformation and contraction of Lie algebras are described in Appendix III.2.2. Let us consider the special class of unimodular Lie algebras. Geometrically this means that on the corresponding connected Lie group G there exists a two-sided invariant top degree differential form, hence, a two-sided invariant measure.

In terms of structure constants the unimodularity condition is given by the equation

$$\operatorname{tr}(\operatorname{ad} X) = 0 \quad \text{for all } X \in \mathfrak{g} \qquad \text{or} \qquad c_{ik}^k := \sum_k c_{ik}^k = 0.$$

We show in Appendix III.2.2 that the structure constants of 3-dimensional unimodular Lie algebras form a 6-dimensional space $\mathbb{R}^6$ which is naturally identified with the space of 3×3 symmetric matrices B with real entries $b^{kl} = b^{lk}$, $k, l = 1, 2, 3$. More precisely, to a matrix $B = \|b^{kl}\|$ there corresponds the collection of structure constants

$$c_{ij}^k = \epsilon_{ijl} b^{kl}.$$

The Lie algebras corresponding to matrices B_1 and B_2 are isomorphic iff there exists a matrix $Q \in GL(3, \mathbb{R})$ such that

$$B_1 = Q^t B_2 Q \cdot \det Q^{-1} \quad \text{where } Q^t \text{ denotes the transposed matrix.}$$

The space $\mathbb{R}^6$ of symmetric matrices splits under these transformations into six $GL(3, \mathbb{R})$-orbits O_k, $1 \leq k \leq 6$. The representatives B_k of these orbits, their dimensions, the corresponding unimodular Lie algebras, and their commutation relations are listed below:

1. $B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\quad \dim O_1 = 6, \quad \mathfrak{g} \simeq \mathfrak{su}(2),$

   $[X, Y] = Z, \quad [Y, Z] = X, \quad [Z, X] = Y.$

2. $B_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\quad \dim O_2 = 6, \quad \mathfrak{g} \simeq \mathfrak{sl}(2, \mathbb{R}),$

   $[X, Y] = Z, \ [Y, Z] = X, \ [Z, X] = -Y.$

3. $B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\quad \dim O_3 = 5, \quad \mathfrak{g} \simeq \mathfrak{so}(2, \mathbb{R}) \ltimes \mathbb{R}^2,$

   $[X, Y] = Z, \ [Y, Z] = 0, \ [Z, X] = Y.$

4. $B_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\quad \dim O_4 = 5, \quad \mathfrak{g} \simeq \mathfrak{so}(1, 1, \mathbb{R}) \ltimes \mathbb{R}^2,$

   $[X, Y] = Z, \ [Y, Z] = 0, \ [Z, X] = -Y.$

5. $B_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\quad \dim O_5 = 4, \quad \mathfrak{g} \simeq \mathfrak{h},$

   $[X, Y] = Z, \ [Y, Z] = [Z, X] = 0.$

6. $B_6 = 0$, $\quad \dim O_6 = 0, \quad \mathfrak{g} \simeq \mathbb{R}^3,$

   $[X, Y] = [Y, Z] = [Z, X] = 0.$

The matrix B of type 5 can be obtained as a limit of matrices of any previous types.

Exercise 7. Find the explicit formulae for the contraction of the Lie algebras $\mathfrak{su}(2, \mathbb{C})$, $\mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{so}(2, \mathbb{R}) \ltimes \mathbb{R}^2$, $\mathfrak{so}(1, 1, \mathbb{R}) \ltimes \mathbb{R}^2$ to the Heisenberg Lie algebra $\mathfrak{h}$.

Answer: In the notation above let $X_\epsilon = \epsilon X$, $Y_\epsilon = \epsilon Y$, $Z_\epsilon = \epsilon^2 Z$ be the variable basis in $\mathfrak{g}$. When $\epsilon \to 0$, the variable commutation relations tend to the limit relations $[X_0, Y_0] = Z_0$, $[Y_0, Z_0] = [X_0, Z_0] = 0$. ♣

We see that the Heisenberg algebra is indeed situated in the core of the set of all 3-dimensional unimodular Lie algebras. In physical applications this fact is intensively used: the representation theory of these Lie algebras and corresponding Lie groups can be constructed as a deformation of the representation theory of the Heisenberg group.

2. Canonical commutation relations

The algebraic structure of the universal enveloping algebra $U(\mathfrak{h})$ is relatively simple. To describe this structure it is convenient to use the formalism of free boson operators from quantum field theory. Here we explain the mathematical (mostly algebraic) content of this formalism.

2.1. Creation and annihilation operators.

A quantum mechanical description of a free boson particle with one degree of freedom involves two self-adjoint unbounded operators: the coordinate (or position) operator q and the momentum (or impulse) operator p, which are subjected to the

Canonical commutation relations (CCR):

$$pq - qp = \frac{h}{2\pi i} \cdot 1 = -i\hbar \cdot 1. \tag{13}$$

Here h is a **Planck constant** and $\hbar := \frac{h}{2\pi}$ is the so-called **normalized Planck constant**.

Note that in physical applications h and $\hbar$ are dimensional quantities. Their dimension is

$$[h] = [\hbar] = \ length^2 \times mass \times time^{-1} \quad \text{or} \quad energy \times time.$$

The numerical value of $\hbar$ is

$$\hbar \approx 10^{-27} \cdot \frac{g \cdot cm^2}{sec} = 10^{-34} \cdot \frac{kg \cdot m^2}{sec}.$$

In ordinary scale it is a very small number, practically zero. This explains why in classical mechanics p and q are considered as commuting quantities.

In quantum field theory the so-called **free boson operators** or, more precisely, the **creation** and **annihilation operators** a^*, a are used instead of p and q.

These operators are defined by the equations

$$a = \frac{1}{\sqrt{2}}(q + ip), \quad a^* = \frac{1}{\sqrt{2}}(q - ip) \tag{14}$$

and satisfy a variant of CCR: $aa^* - a^*a = \hbar \cdot 1$.

In mathematical physics a system of units where $\hbar = 1$ is mostly used, so that the commutation relation between a and a^* simply becomes

$$aa^* - a^*a = 1. \tag{15}$$

It turns out that many important operators can be written as polynomials or power series in a and a^* and their properties can be studied in a purely algebraic way using only the commutation relation (15). We shall see some examples of this below.

Let us denote by W_n the so-called **Weyl algebra**: the associative algebra with unit over $\mathbb{C}$ generated by n pairs of operators a_k, a_k^* satisfying the relations

$$a_k a_j = a_j a_k, \qquad a_k^* a_j^* = a_j^* a_k^*, \qquad a_k a_j^* - a_j^* a_k = \delta_{kj}. \tag{16}$$

Sometimes, instead of creation and annihilation operators a_k^*, a_k, we shall use other generators $p_k = \frac{i}{\sqrt{2}}(a_k^* - a_k)$ and $q_k = \frac{1}{\sqrt{2}}(a_k^* + a_k)$ with commutation relations

$$p_j p_k = p_k p_j, \qquad q_j q_k = q_k q_j, \qquad p_j q_k - q_k p_j = -i\delta_{kj}. \tag{16'}$$

The algebra W_n can be realized as the algebra of all differential operators with polynomial coefficients in n variables $z_1, \ldots, z_n$. The generators in this realization have the form

$$a_k^* = z_k \text{ (multiplication)}, \qquad a_k = \partial/\partial z_k \text{ (differentiation)}.$$

The space of polynomials $\mathbb{C}[z_1, \ldots, z_n]$ admits a special scalar product such that a_k^* is adjoint to a_k with respect to this product. Namely:

$$(P, Q) = \int_{\mathbb{C}^n} P(z)\overline{Q(z)} e^{-|z|^2} |d^n z|^2$$

where $|z|^2 = \sum_k z_k \overline{z_k}$ and $|d^n z|^2 = |dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n|$, $z_k = x_k + iy_k$.

2.2. Two-sided ideals in $U(\mathfrak{h})$.

The structure of $U(\mathfrak{h})$ is described in

Theorem 1. a) *The center $Z(\mathfrak{h})$ of $U(\mathfrak{h})$ is generated by the central basic element $Z \in \mathfrak{h}$, i.e. $Z(\mathfrak{h}) = \mathbb{C}[Z]$.*

b) *Any non-zero two-sided ideal in $U(\mathfrak{h})$ has a non-zero intersection with $Z(\mathfrak{h})$.*

c) *The maximal two-sided ideals in $U(\mathfrak{h})$ have the form*

$$I_\lambda = (Z - \lambda \cdot 1)U(\mathfrak{h})$$

and for any $\lambda \neq 0$ the quotient algebra $U(\mathfrak{h})/I_\lambda$ is isomorphic to the Weyl algebra W_1.

Proof. a) We can use any of the maps $a^{-1}, b^{-1}, \mathbf{sym}^{-1}$ to identify elements of $U(\mathfrak{h})$ with ordinary polynomials in x, y, z. In all three cases the commutators with basic vectors X, Y in this realization act as differential operators:

$$\operatorname{ad} X = z\partial/\partial y, \qquad \operatorname{ad} Y = -z\partial/\partial x. \tag{17}$$

Therefore the central elements correspond to those polynomials $P(x, y, z)$ that satisfy $\partial P/\partial x = \partial P/\partial y = 0$. Hence, $P = P(z)$ and a) is proved.

b) It is clear that starting with any non-zero polynomial $P(x, y, z)$ and applying the operators (17) several times we eventually come to a non-zero polynomial in z. This proves b).

c) Now let $I \subset U(\mathfrak{h})$ be a maximal two-sided ideal. Then its intersection with $Z(\mathfrak{h})$ is a maximal ideal in $Z(\mathfrak{h}) \simeq \mathbb{C}[Z]$. Hence,

$$I \cap Z(\mathfrak{h}) = (Z - \lambda \cdot 1)\mathbb{C}[Z]$$

and $I \supset I_\lambda$. But the quotient algebra $U(\mathfrak{h})/I_\lambda$ for $\lambda \neq 0$ is clearly isomorphic to the Weyl algebra W_1. Since W_1 has no non-trivial ideals, $I = I_\lambda$ and we are done. □

2.3. H. Weyl reformulation of CCR.

The physical formulation of CCR given above is short and computationally convenient but not quite satisfactory from the mathematical point of view. The point is that the product of two unbounded operators is not well defined,[1] so that (13) and (15) make no sense in general.

We discuss here the correct reformulation of CCR due to H. Weyl.

[1] More precisely, the domain of definition for the product can be the zero subspace $\{0\}$.

According to the M. Stone theorem (see Appendix V.1.2), for any self-adjoint, not necessarily bounded, operator A in a Hilbert space H the operator iA is a generator of a one-parametric group $U(t)$ of unitary operators:

$$U(t) = e^{itA}, \qquad A = \frac{1}{i}\frac{d}{dt}U(t)\Big|_{t=0}.$$

If the operators p and q satisfy CCR, then the formal computation gives $pq^n = q^n p + \frac{h}{2\pi i} n q^{n-1}$. It follows that for any polynomial F we have

$$pF(q) = F(q)p + \frac{h}{2\pi i}F'(q).$$

Extending this relation to power series in q, we get

$$pe^{itq} = e^{itq}(p + \hbar t) \qquad \text{or} \qquad e^{-itq}p\,e^{itq} = p + \hbar t.$$

This implies

$$e^{-itq}F(p)e^{itq} = F(p + \hbar t)$$

and, finally,

$$e^{-itq}e^{isp}e^{itq} = e^{is(p+\hbar t)}.$$

Let us introduce the notation $u(s) := e^{isp}$, $v(t) := e^{itq}$. Then the above relation takes the form

$$u(s)v(t) = e^{ist\hbar}v(t)u(s). \tag{18}$$

We have seen that (18) is a formal consequence of (13). Conversely, one can easily obtain (13) by differentiating both sides of (18) with respect to s and t. So, on the formal level, (13) and (18) are equivalent.

But there is an essential difference between these two relations. Namely, the latter relation involves only bounded (actually, unitary) operators. So, there is no questions about the domain of definition.

We formulate now the **Weyl form** of CCR:

Two self-adjoint operators p and q satisfy CCR if the corresponding 1-*parametric groups of unitary operators $u(s) = e^{isp}$ and $v(t) = e^{itq}$ satisfy the relation* (18).

Although the Weyl form (18) of CCR has the advantage of mathematical rigor, the original, Heisenberg form (13) of CCR looks simpler and easier to use. Therefore there were many attempts to find some additional conditions on the operators p and q that together with (13) would guarantee (18). We formulate here the most convenient one found by E. Nelson.

Nelson condition:

There is a dense subspace $\mathcal{D} \subset H$ which is stable under p and q such that the energy operator $E := (p^2 + q^2)\big|_{\mathcal{D}} = (a^*a + a^*a)\big|_{\mathcal{D}}$ is essentially self-adjoint.

Theorem 2 (see [**Nel**]). *If the self-adjoint operators p and q satisfy CCR* (13) *and the Nelson condition above, then they satisfy* (18). $\square$

Remark 1. Consider four operators $X := ip$, $Y := iq$, $Z := i\hbar\cdot 1$, $T := \frac{1}{2i\hbar}E$. They span a so-called **diamond Lie algebra** $\mathfrak{d}$ with commutation relations

$$[T, X] = Y, \qquad [T, Y] = -X, \qquad [X, Y] = Z, \qquad [Z, \mathfrak{d}] = 0.$$

Note that the first three basic vectors span the ideal in $\mathfrak{d}$ isomorphic to the Heisenberg Lie algebra $\mathfrak{h}$. The Nelson condition can be reformulated in representation-theoretic terms. Namely, we require that the given representation of the Heisenberg algebra $\mathfrak{h}$ can be extended to a representation of the diamond Lie algebra $\mathfrak{d} \supset \mathfrak{h}$ by skew self-adjoint operators in H. $\heartsuit$

2.4. The standard realization of CCR.

The standard realization of CCR in the complex Hilbert space $L^2_{\mathbb{C}}(\mathbb{R}, dt)$ is

$$p = -i\hbar\frac{d}{dt}, \qquad q = t. \tag{19}$$

Here we assume that p and q have the so-called **natural domain of definition** (see the remark below).

Remark 2. For a differential operator $A = \sum_k a_k(t)\frac{d^k}{dt^k}$ with smooth coefficients the natural domain of definition $\mathcal{D}_A$ consists of those functions $f \in L^2_{\mathbb{C}}(\mathbb{R}, dt)$ for which there exists a function $g \in L^2_{\mathbb{C}}(\mathbb{R}, dt)$ such that $Af = g$ in the distributional sense.

It means that for any test function $\phi \in \mathcal{A}(\mathbb{R})$ we have

$$\int_{\mathbb{R}} g(t)\phi(t)dt = \int_{\mathbb{R}} f(t)A^*\phi(t)dt$$

where $A^* = \sum_k (-1)^k \frac{d^k}{dt^k} \circ \overline{a_k(t)}$ is the formal adjoint to A. $\heartsuit$

In our case $\mathcal{D}_q$ consists of those functions $f \in L^2_{\mathbb{C}}(\mathbb{R}, dt)$ for which the function $tf(t)$ belongs to $L^2_{\mathbb{C}}(\mathbb{R}, dt)$. The domain $\mathcal{D}_p$ consists of absolutely continuous complex functions $f \in L^2_{\mathbb{C}}(\mathbb{R}, dt)$ with the derivative $f' \in L^2_{\mathbb{C}}(\mathbb{R}, dt)$.

Exercise 8. Show that both operators p and q with the natural domains of definition are self-adjoint.

Hint. For q it can be checked directly by the definition. For p the direct proof is also possible, but it is simpler to make the Fourier transform that sends the operator p to $-\hbar q$. $\clubsuit$

The creation, annihilation, and energy operators for $\hbar = 1$ have the form

$$(20) \qquad a^* = \frac{1}{\sqrt{2}}\left(t - \frac{d}{dt}\right), \qquad a = \frac{1}{\sqrt{2}}\left(t + \frac{d}{dt}\right), \qquad E = -\frac{d^2}{dt^2} + t^2.$$

Exercise 9.* Check that the Nelson condition is satisfied for the operators above.

Hint. Take $\mathcal{D} = \mathcal{S}(\mathbb{R})$, the Schwartz space. It is certainly stable under both operators p and q. Check that E with the domain $\mathcal{D}$ is essentially self-adjoint using the criterion from Appendix IV.2.3. ♣

We say that a bounded operator B commutes with an unbounded operator A if B preserves the domain of definition D_A and the equality $AB = BA$ holds in D_A.

Equivalent formulation: B commutes with all bounded functions of A (actually, it is enough to check that B commutes with the spectral function $E_c(A)$ for all $c \in \mathbb{R}$).

Theorem 3. *The standard realization of CCR is operator irreducible, i.e. every bounded operator A which commutes with p and q is a scalar operator.*

Proof. We start with a result which is an infinite-dimensional analogue of the following well-known fact from linear algebra:

If a diagonal matrix T has different eigenvalues, then any matrix A that commutes with T is itself a diagonal matrix.

This result is formulated as follows:

Lemma 1. *Any bounded operator A in $L^2_{\mathbb{C}}(\mathbb{R}, dt)$ which commutes with multiplication by t is an operator of multiplication by some (essentially bounded) complex function $a(t)$.*

Proof. Consider the subset $H_0 \subset L^2_{\mathbb{C}}(\mathbb{R}, dt)$ consisting of functions of the form $f(t) = P(t)e^{-t^2}$ for some polynomial $P(t)$. We have

$$A\,f(t) = A\,P(t)e^{-t^2} = A \circ P(q)\,e^{-t^2} = P(q) \circ A\,e^{-t^2} = a(t)f(t)$$

where $a(t) = e^{t^2} A\,e^{-t^2}$. So, A is indeed the multiplication by $a(t)$ on the subset H_0. Since H_0 is dense[2] in $L^2_{\mathbb{C}}(\mathbb{R}, dt)$, for any operator B the norm of the restriction $B|_{H_0}$ is equal to $\|B\|$.

Therefore, the function $a(t)$ must be essentially bounded and the operator A acts as the multiplication by $a(t)$ everywhere. □

[2]For the proof of this non-trivial statement, see, e.g., [**KG**]. Cf. also the proof of the Stone–von Neumann uniqueness theorem in Section 2.6.

Now, we use the fact that A commutes with p. On the subset of smooth functions from $L^2_{\mathbb{C}}(\mathbb{R}, dt)$ we have

$$\frac{1}{2\pi i}[a(t)f(t)]' = p \circ A\, f = A \circ p\, f = \frac{1}{2\pi i}a(t)f'(t).$$

It follows that $a(t)$ has a zero generalized derivative, hence is a constant almost everywhere. □

We finish our description of the standard realization by introducing the operator $N = a^*a$. In the standard realization we have $N = \frac{1}{2}(p^2 + q^2 - 1)$.

The physical meaning of this operator is the number of particles: the eigenvector v_k of N corresponding to the eigenvalue $k \in \mathbb{N}$ describes the k-particle state of the system. In particular, the vector v_0 corresponds to the 0-particle state and is called a **vacuum vector**. It can also be defined as a unit vector satisfying $av = 0$.

It is easy to check that the operator N satisfies the commutation relations

(21)
$$[N, a] = -a, \quad [N, a^*] = a^*, \quad \text{or} \quad Na = a(N-1), \quad Na^* = a^*(N+1).$$

It follows from (21) that if $v \in \mathcal{H}$ is an eigenvector for N with an eigenvalue k, then av and a^*v are also eigenvectors for N with eigenvalues $k \mp 1$. This property justifies the terminology of creating and annihilating operators.

In the standard realization the vacuum vector is the function $v_0 \in L^2_{\mathbb{C}}(\mathbb{R}, dt)$ satisfying the equation $av_0 = 0$, or $\hbar v_0' = -tv_0$. It is unique up to a phase factor:

$$v_0(t) = \frac{e^{i\alpha}}{\sqrt{2\pi\hbar}} \cdot e^{-\frac{t^2}{2\hbar}}. \tag{22}$$

We shall continue the study of N later.

2.5. Other realizations of CCR.

Although all Hilbert spaces of Hilbert dimension $\aleph_0$ are isomorphic, the concrete realizations can be very different. Here we describe the realization $\mathcal{H}$ consisting of holomorphic functions. It has the following advantage: elements of $\mathcal{H}$ are genuine functions, not equivalence classes as in the realization $H = L^2_{\mathbb{C}}(\mathbb{R}, dt)$.

Consider the complex plane $\mathbb{C}$ with coordinate $z = x + iy$. Endow it with a normalized measure

$$\mu = \frac{1}{\pi}e^{-(x^2+y^2)}dxdy.$$

Let $\mathcal{H}$ be the set of all holomorphic functions f on $\mathbb{C}$ that belong to $L^2(\mathbb{C},\,\mu)$. We list the main properties of $\mathcal{H}$ in

Proposition 2. a) *$\mathcal{H}$ is closed in $L^2(\mathbb{C},\,\mu)$, hence is a Hilbert space.*

b) *The space of polynomials $\mathbb{C}[z]$ is dense in $\mathcal{H}$.*

c) *The monomials $\phi_k(z) = \frac{z^k}{\sqrt{k!}}$, $k \geq 0$, form an orthonormal basis in $\mathcal{H}$.*

d) *The evaluation functional $F_c(f) = f(c)$, $c \in \mathbb{C}$, is continuous on $\mathcal{H}$ and can be written as a scalar product:*

$$F_c(f) = (f,\, e_c) \qquad \text{where } e_c \in \mathcal{H} \text{ is given by } e_c(z) = e^{\bar{c}z}. \tag{23}$$

e) *The functions e_c, $c \in \mathbb{C}$, form a* **continuous basis** *in $\mathcal{H}$ in the following sense: for any $f \in \mathcal{H}$ we have*

$$f(z) = \int_{\mathbb{C}} f(c)e_c(z)d\mu(c) = \int_{\mathbb{C}} (f,\, e_c)e_c(z)d\mu(c). \tag{24}$$

Proof. a) Assume that a sequence of holomorphic functions $\{f_n\}$ converges in $L^2(\mathbb{C},\,\mu)$. We show that it also converges uniformly on bounded sets. Indeed, holomorphic functions have the **mean value property**:

$$f(a) = \frac{1}{\pi r^2}\int_{|z-a|\leq r} f(z)dxdy \qquad \text{for any } a \in \mathbb{C},\ r > 0.$$

It follows that the value $f(a)$ can be written as a scalar product (f, f_a) for some $f_a \in L^2(\mathbb{C},\,\mu)$. For example, we can take

$$f_a(z) = \frac{e^{|z|^2}}{r^2}\chi_{r,a}(z)$$

where $\chi_{r,a}$ is the characteristic function of the disc $|z-a| \leq r$.

It is clear that when a runs through a bounded set in $\mathbb{C}$, the functions f_a (for a fixed $r > 0$) form a bounded set in $L^2(\mathbb{C},\,\mu)$. Hence, $f_n(a) \to f(a)$ uniformly on bounded sets.

Since a uniform limit of holomorphic functions is holomorphic, we see that $\mathcal{H}$ is closed in $L^2(\mathbb{C},\,\mu)$.

b) First, we note that the $\{\phi_k\}$ form an orthonormal system in $\mathcal{H}$. The proof follows from the direct computation:

$$(\phi_k,\, \phi_l) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{z^k\bar{z}^l}{\sqrt{k!\,l!}}e^{-z\bar{z}}dxdy = \delta_{kl}\int_{r\geq 0}\frac{r^{2k}}{k!}e^{-r^2}dr^2 = \delta_{kl}.$$

In the case where f is a holomorphic function in $L^2(\mathbb{C}, \mu)$, its Taylor series $\sum_{n\geq 0} f^{(n)}(0)\frac{z^n}{n!}$ converges both pointwise and in the norm of $L^2(\mathbb{C}, \mu)$. (The latter follows from the Bessel inequality.)

The pointwise limit is obviously $f(z)$. Consider the L^2-limit. According to a), it is a holomorphic function in $L^2(\mathbb{C}, \mu)$. Also, it has the same coefficients with respect to the orthonormal system $\{\phi_k\}_{k\geq 0}$ as f. But these coefficients up to a factor are just the Taylor coefficients. Hence, the L^2-limit is also equal to f.

c) From b) it follows that the system $\{\phi_k\}$ is complete, hence, is a Hilbert basis.

d) Let us check that (23) is true for $f = \phi_k$. Indeed,

$$(\phi_k, e_c) = \left(\phi_k, \sum_{n\geq 0} \frac{\overline{c}^n z^n}{n!}\right) = \left(\phi_k, \sum_{n\geq 0} \frac{\overline{c}^n \phi_n}{\sqrt{n!}}\right) = \sum_{n\geq 0} \delta_{kn} \frac{c^n}{\sqrt{n!}} = \phi_k(c).$$

By linearity, the equality holds for any polynomial function f. Finally, the argument we used to prove a) and b) shows that (23) holds for any $f \in \mathcal{H}$.

e) Compare the coefficients of both sides with respect to the basis $\{\phi_k\}$, $k \geq 0$. The right-hand side gives

$$\int_{\mathbb{C}} f(c)(e_c, \phi_k)d\mu(c) = \int_{\mathbb{C}} f(c)\frac{\overline{c}^k}{\sqrt{k!}}d\mu(c) = (f, \phi_k),$$

which is exactly the left-hand side. □

The remarkable feature of $\mathcal{H}$ is that the operators of multiplication by z and differentiation with respect to z are conjugate:

$$(zf_1, f_2) = \left(f_1, \frac{\partial}{\partial z} f_2\right). \tag{25}$$

Indeed, integrating by parts, we get

$$(f_1, f_2') = \frac{1}{\pi}\int_{\mathbb{C}} f_1(z)\overline{f_2'(z)}e^{-z\overline{z}}dxdy = \frac{1}{\pi}\int_{\mathbb{C}} e^{-z\overline{z}} f_1(z)\left(\overline{f_2(z)}\right)'_{\overline{z}} dxdy$$
$$= -\frac{1}{\pi}\int_{\mathbb{C}} \overline{f_2(z)}\left(e^{-z\overline{z}}f_1(z)\right)'_{\overline{z}} dxdy = \frac{1}{\pi}\int_{\mathbb{C}} zf_1(z)\overline{f_2(z)}e^{-z\overline{z}}dxdy = (zf_1, f_2).$$

Moreover, we have the obvious relation $[\frac{\partial}{\partial z}, z] = 1$. Therefore, we obtain a realization of CCR in $\mathcal{H}$ putting

$$a = \frac{\partial}{\partial z}, \qquad a^* = z. \tag{26}$$

It is called the **Fock realization**. The vacuum vector in this realization is $v_0(z) \equiv 1$ and the k-particle state is $v_k(z) = \frac{z^k}{\sqrt{k!}}$.

Exercise 10. Establish an explicit correspondence between the Fock realization and the standard one.

Answer: The desired maps $A : \mathcal{H} \to L^2(\mathbb{R}, dt)$ and A^{-1} are given by

$$(Af)(t) = \int_{\mathbb{C}} f(z) e^{-\frac{t^2}{2} + t\bar{z}\sqrt{2} - \frac{\bar{z}^2}{2}} d\mu, \quad (A^{-1}\phi)(z) = \int_{\mathbb{R}} \phi(t) e^{-\frac{t^2}{2} + tz\sqrt{2} - \frac{z^2}{2}} dt.$$

♣

There is one more remarkable realization of CCR. Its existence is related to the following fact. Put $u = e^{2\pi i h^{-1} p}$ and $v = e^{2\pi i q}$. From (23) we see that u and v commute. It turns out that the Laurent polynomials $\sum_{m,\, n \in \mathbb{Z}} c_{m,n} u^m v^n$ form a maximal commutative subalgebra, isomorphic to the algebra of smooth functions on the 2-dimensional torus $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$.

The realization in question acts on the Hilbert space $\Gamma^2(L)$ of square integrable sections of a certain complex line bundle $\mathbb{C} \to L \xrightarrow{p} \mathbb{T}^2$.

To describe it, we start with the space $\Gamma(L)$ of smooth sections. According to the algebraic definition of a line bundle (see Appendix II.2.1), $\Gamma(L)$ is a module of local rank 1 over the algebra $\mathcal{A}(\mathbb{T}^2)$. We identify $\mathcal{A}(\mathbb{T}^2)$ with the algebra of all smooth double-periodic functions on $\mathbb{R}^2$. The space $\Gamma(L)$ consists of all smooth functions ϕ on $\mathbb{R}^2$ satisfying

$$\phi(\tau + m,\, \theta + n) = \phi(\tau,\, \theta) e^{2\pi i m \theta} \quad \text{for all } m,\, n \in \mathbb{Z}. \tag{27}$$

It is evidently the module over the algebra of double-periodic functions.

It turns out that $\Gamma(L)$ can be naturally identified with the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing smooth functions on $\mathbb{R}$.

Proposition 3. *The equations*

$$\phi(\tau,\, \theta) = \sum_{k \in \mathbb{Z}} f(\tau + k) e^{-2\pi i k \theta} \qquad \textit{and} \qquad f(t) = \int_0^1 \phi(t,\, \theta) d\theta \tag{28}$$

establish the reciprocal bijections $\phi \leftrightarrow f$ between $\Gamma(L)$ and $\mathcal{S}(\mathbb{R})$.

Proof. First, we observe that for any $f \in \mathcal{S}(\mathbb{R})$ the series in (28) converges uniformly together with all derivatives. It follows that this series defines a smooth function $\phi(\tau,\, \theta)$, which satisfies (27).

Conversely, for any $\phi \in \Gamma(L)$ the integral in the second part of (28) defines a smooth function f on $\mathbb{R}$. We want to show that this function belongs to $\mathcal{S}(\mathbb{R})$, i.e. that

$$\max_{t \in \mathbb{R}} |t^m f^{(l)}(t)| < \infty \quad \text{for all } k,\, l \in \mathbb{Z}_+.$$

From (27) and the second part of (28) we obtain

$$f(t+k) = \int_0^1 \phi(t,\,\theta) e^{2\pi i k\theta} d\theta.$$

Therefore

$$|(2\pi k)^m f^{(l)}(t+k)| \leq \int_0^1 |\partial_t^l \partial_\theta^m \phi(t,\,\theta)| d\theta =: c_{m,l}.$$

Now, if we write t in the form $t = k + \{t\}$ where $k \in \mathbb{Z}$ and $\{t\} \in [0,1)$, we obtain

$$|t^m f^{(l)}(t)| < (|k|+1)^m |f^{(l)}(k+\{t\})| < \sum_{s=0}^{m} \binom{m}{s} \frac{c_{s,l}}{(2\pi)^s} < \infty.$$

Finally, the two maps (28) are reciprocal because

$$\int_0^1 e^{2\pi i k\theta} d\theta = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

□

The operators $q = t$ and $p = \frac{h}{2\pi i}\frac{d}{dt}$ act on $\mathcal{S}(\mathbb{R})$. After the identification above they take the form

$$\widetilde{q} = \tau - \frac{1}{2\pi i}\,\partial_\theta, \qquad \widetilde{p} = \frac{h}{2\pi i}\,\partial_\tau. \tag{29}$$

Define the scalar product of two smooth sections of L by

$$(\phi_1,\,\phi_2) = \int_0^1 \int_0^1 \phi_1(\tau,\,\theta)\overline{\phi_2(\tau,\,\theta)} d\tau d\theta. \tag{30}$$

(Note that because of (27) the integrand above is a periodic function on $\mathbb{R}^2$, so the integration is actually over the torus $\mathbb{T}^2$.)

It is not difficult to see that the operators (29) initially defined on the space of smooth sections are essentially self-adjoint with respect to the scalar product (30). Moreover, they evidently satisfy CCR.

The computation of the vacuum vector in this realization leads to very interesting number-theoretic investigations, which we leave aside (see **[LV]** for details).

2.6. Uniqueness theorem.

One very important fact is that the standard realization of CCR is actually the unique one. We give here three formulations and outline three different proofs of this theorem.

Theorem 4 (Stone–von Neumann uniqueness theorem). *Let p and q be self-adjoint operators in a Hilbert space H satisfying the canonical commutation relation in the Heisenberg form* (13) *and also the Nelson condition.*

Then H is isomorphic to a direct sum of several copies of $L^2(\mathbb{R}, dt)$ where the operators p and q act by the standard realization (19).

In particular, any irreducible realization of CCR is equivalent to the standard one.

Proof. The idea is to investigate in detail the spectrum of the self-adjoint operator $N = a^*a$.

Proposition 4. a) *In any realization of CCR there exists a vacuum vector.*

b) *For an irreducible realization the vacuum vector is unique (up to a scalar factor).*

c) *All irreducible realizations of CCR are equivalent to each other. There exists an orthonormal basis $\{v_k\}_{k\geq 0}$ such that the operators a, N, a^* are given by*

$$(31) \qquad av_k = \sqrt{k}\cdot v_{k-1}, \qquad Nv_k = k\cdot v_k, \qquad a^*v_k = \sqrt{k+1}\cdot v_{k+1}.$$

Proof of Proposition 4. a) Observe that N is a non-negative operator:

$$(Nv, v) = |av|^2 \geq 0.$$

So, the spectrum of N is contained in the right half-line.

Let E_c, $c \in \mathbb{R}$, be the spectral function for N. From (15) we conclude that $E_c a = aE_{c+1}$.[3] Let $c_0 \geq 0$ be the minimal point of the spectrum of N. Then for any $\epsilon \in (0, 1)$ the projector $E_{c_0+\epsilon}$ is non-zero, while $aE_{c_0+\epsilon} = E_{c_0+\epsilon-1}a = 0$. Therefore, all vectors from the subspace $H_{c_0+\epsilon} := E_{c_0+\epsilon}H$ are annihilated by a. Hence, they are vacuum vectors (eigenvectors for N with zero eigenvalue).

We see that actually $c_0 = 0$ and the zero eigenspace for N is different from $\{0\}$. Thus, any realization of CCR has a vacuum vector.

b) and c). Choose an arbitrary vacuum vector v_0 of unit length and put

$$(32) \qquad v_k := \frac{(a^*)^k}{\sqrt{k!}}v_0.$$

[3]Here and below we use the fact that all vectors in the range of E_c belong to the domain of definition of the operators a and a^*. It follows easily from the study of the graph of N (see Appendix IV.2.3).

Then the last two equalities (31) will be satisfied. Let us prove the first one. Indeed,

$$av_k = \frac{1}{\sqrt{k!}}a(a^*)^k v_0 = \frac{1}{\sqrt{k!}}[a,\,(a^*)^k]v_0 = \frac{k}{\sqrt{k!}}(a^*)^{k-1}v_0 = \sqrt{k}\cdot v_{k-1}.$$

The vectors v_k are orthogonal since they belong to different eigenspaces of N. From the equalities

$$|v_k|^2 = (v_k,\, k^{-\frac{1}{2}}a^* v_{k-1}) = (k^{-\frac{1}{2}}av_k,\, v_{k-1}) = |v_{k-1}|^2$$

we see that $|v_k| = 1$ for all $k \geq 0$. Therefore, $\{v_k\}$ forms an orthonormal system.

Suppose now that the realization in question is irreducible. According to (31), the Hilbert subspace spanned by $\{v_k\}_{k\geq 0}$ is invariant under both operators a and a^*. Then it must coincide with the whole space.

The uniqueness of the vacuum vector in H now follows immediately from (31): for any vector $v = \sum_{k\geq 0} c_k v_k$ we have $av = \sum_{k\geq 1} c_k\sqrt{k}v_{k-1}$. So, $av = 0$ iff $c_k = 0$ for $k \geq 1$, i.e. $v = c_0 v_0$. □

We return to the proof of Theorem 4. Consider an arbitrary realization of CCR in a Hilbert space H. Let H_0 be the subspace of vacuum vectors. Let $\{v_0^{(\alpha)}\}_{\alpha\in A}$ be an orthonormal basis in H_0 and put $v_k^{(\alpha)} = \frac{(a^*)^k}{\sqrt{k!}}v_0^{(\alpha)}$. Let $H^{(\alpha)}$ be the Hilbert space spanned by $\{v_k^{(\alpha)}\}_{k\geq 0}$. We leave to the reader to check that $H = \boxplus_{\alpha\in A}H^{(\alpha)}$ (direct sum in the category of Hilbert spaces) and in each $H^{(\alpha)}$ we have the standard realization of CCR. □

Note that in the course of the demonstration we obtained another proof of the irreducibility of the standard realization. Indeed, we have seen that in the standard realization we have a unique vacuum vector (22).

The explicit formula for the functions v_k can be obtain using the relation

$$a^* = e^{\frac{t^2}{2}} \circ -\frac{1}{\sqrt{2}}\frac{d}{dt} \circ e^{-\frac{t^2}{2}}.$$

Namely,

$$v_k = \frac{(a^*)^k}{\sqrt{k!}}v_0 = \frac{e^{\frac{t^2}{2}}}{\sqrt{k!}}\left(-\frac{1}{\sqrt{2}}\frac{d}{dt}\right)^k e^{-t^2} = \frac{H_k(t)}{\sqrt{k!}}e^{-\frac{t^2}{2}}$$

where H_k are the so-called **Hermite polynomials**. We recover the well-known fact that functions v_k, called **Hermite functions**, form an orthonormal basis in $L^2(\mathbb{R},\, \frac{dt}{\pi})$.

The second formulation and proof of the uniqueness theorem uses the Weyl form of CCR directly.

Theorem 4′. *Let H be a Hilbert space, and let $u(s) = e^{isp}$, $v(t) = e^{itq}$ be the two 1-parametric groups of unitary operators in H that satisfy the CCR relation in the Weyl form (18). Assume also that the space H is irreducible with respect to these groups.*

Then there is an isomorphism $\alpha : H \to L^2(\mathbb{R}, dx)$ such that for all s and t the following diagrams are commutative:

$$\begin{array}{ccc} H & \xrightarrow{u(s)} & H \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ L^2(\mathbb{R}, dx) & \xrightarrow{U(s)} & L^2(\mathbb{R}, dx), \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{v(t)} & H \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ L^2(\mathbb{R}, dx) & \xrightarrow{V(t)} & L^2(\mathbb{R}, dx) \end{array}$$

where

$$(U(s)f)(x) = f(x + \hbar s), \qquad (V(t)f)(x) = e^{itx} f(x). \tag{33}$$

Proof. We shall use the spectral theorem for the self-adjoint operator $q = -iv'(0)$, the generator of the group $\{v(s)\}$ (see Appendix IV.2.4). According to this theorem, there exists a family $(\mu_1, \mu_2, \ldots, \mu_\infty)$ of pairwise disjoint Borel measures on $\mathbb{R}$ such that H is isomorphic to the direct sum of one copy of $L^2(\mathbb{R}, \mu_1)$, two copies of $L^2(\mathbb{R}, \mu_2)$, ..., and a countable set of copies of $L^2(\mathbb{R}, \mu_\infty)$ where q acts as multiplication by x. Moreover, these measures are defined uniquely up to equivalence.

The relation (18) implies that $u(s)qu(s)^{-1} = q + \hbar s \cdot 1$. So, the operators q and $q + a \cdot 1$ are unitarily equivalent for all $a \in \mathbb{R}$. But the spectral measures $\{\mu_k\}$ for the operator $q + a \cdot 1$ are just the spectral measures $\{\mu_k\}$ for q shifted by a. It follows that each measure μ_k is equivalent to the shifted measure. Such measures are called **quasi-invariant** under the shifts. We now use the following fact from measure theory.

Lemma 2. *Any non-zero Borel measure μ on $\mathbb{R}$, which is quasi-invariant under all shifts, is equivalent to the Lebesgue measure $\lambda = |dx|$.*

Proof of Lemma 2. Consider the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ with the product measure $\lambda \times \mu$. For any Borel subset $A \subset \mathbb{R}$ denote by X_A the subset of the plane defined by

$$X_A = \{(x, y) \in \mathbb{R}^2 \mid y - x \in A\}.$$

We can compute the $(\lambda \times \mu)$-measure of X_A in two different ways using Fubini's theorem:

1. $(\lambda \times \mu)(X_A) = \int_{X_A} \lambda \times \mu = \int_{\mathbb{R}} \lambda(y - A) d\mu(y) = \lambda(A)\mu(\mathbb{R})$.
2. $(\lambda \times \mu)(X_A) = \int_{X_A} \mu \times \lambda = \int_{\mathbb{R}} \mu(x + A) dx = \begin{cases} 0 & \text{if } \mu(A) = 0, \\ > 0 & \text{if } \mu(A) > 0. \end{cases}$

We see that the conditions $\mu(A) = 0$ and $\lambda(A) = 0$ are equivalent. Hence, the measure μ is equivalent to λ. $\square$

We return to the proof of Theorem 4′. The measures μ_k are pairwise disjoint and at the same time equivalent to λ, if non-zero. We conclude that all μ_k vanish except for some μ_n which is equivalent to λ.

Hence, we can assume that H is a direct sum of n copies of $L^2(\mathbb{R}, dx)$ where q acts as the multiplication by x. The space H can be interpreted in two ways: either as a tensor product $W \otimes L^2(\mathbb{R}, dx)$ where W is an n-dimensional Hilbert space, or as the space $L^2(\mathbb{R}, W, dx)$ of W-valued functions on $\mathbb{R}$. We shall use both interpretations.

Thus, we have reduced the operators $v(t)$ to the canonical form

$$v(t) = 1 \otimes V(t) \tag{34}$$

where the $V(t)$ are given in (33).

The next step is reducing the operator p to the form $\frac{h}{2\pi i}\frac{d}{dx}$, or, what is the same, reducing $u(s)$ to the form $1 \otimes U(s)$. To this end we observe that $u(s)$ and $1 \otimes U(s)$ interact in the same way with $v(t)$:

$$u(s)v(t)u(s)^{-1} = e^{i\hbar st}v(t) = (1 \otimes U(s)) \cdot v(t) \cdot (1 \otimes U(s))^{-1}.$$

Indeed, both equalities follows from (18) if we recall that $v(t) = 1 \otimes V(t)$. Therefore, the operator $u(s)^{-1}(1 \otimes U(s))$ commutes with $v(t)$ for all t.

Now, we use the following general fact.

Lemma 3. *Any bounded operator in $L^2(\mathbb{R}, W, dx)$ that commutes with multiplication by $e^{itx}, t \in \mathbb{R}$, is itself a multiplication by an operator-valued function $A(x)$ with values in* $\operatorname{End} W$.

Proof of Lemma 3. For $k = 1$ the statement is similar to Lemma 1 in Section 2.4 and can be proved in the same way. Namely, we consider the subspace $H_0 \subset L_2(\mathbb{R}, dx)$ that consists of functions of the form $f(x) = T(x)e^{-x^2}$ where $T(x)$ is a trigonometric polynomial (a linear combination of functions e^{itx} for different t's). This subspace is dense in $L_2(\mathbb{R}, dx)$ and an operator A that commutes with multiplication by all e^{itx}, $t \in \mathbb{R}$, on the subspace H_0 coincides with multiplication by the function $a(x) = e^{x^2} A\, e^{-x^2}$.

For general k we proceed as follows. Let $w_1, \ldots, w_n$ be any orthonormal basis in W. Introduce the functions

$$a_{ij}(x) = e^{x^2}(A(e^{-x^2} \otimes w_j), w_i)_W, \quad 1 \le i, j \le k.$$

It is straightforward that on the subspace $H_0 \otimes W$ the operator A coincides with the multiplication by the matrix-function $A(x) := \|a_{ij}(x)\|$. Since

$H_0 \otimes W$ is dense in H, the norm of $A(x)$ is almost everywhere bounded by $\|A\|$. Hence, the multiplication by $A(x)$ defines a bounded operator on H. This operator coincides with A on $H_0 \otimes W$, hence everywhere. □

Applying Lemma 3 to the operators $u(s)^{-1}(1 \otimes U(s))$, we reduce $u(s)$ to the special form

$$u(s) = A(s, x) \otimes U(s) \tag{35}$$

where for all $s \in \mathbb{R}$ the matrix $A(s, x)$ defines a unitary operator in W for almost all $x \in \mathbb{R}$.

Since $u(s)$ as well as $U(s)$ form 1-parametric groups, we get the relations

$$A(s_1, x)A(s_2, x + s_1) = A(s_1 + s_2, x), \quad A(-s, x + s) = A(s, x)^{-1},$$

which hold for any s_1, s_2, s for almost all x.

Let us put $B(x, y) := A(y - x, x)$, $s_1 = y - x$, $s_2 = z - y$. Then $A(s, x) = B(x, x + s)$ and the above relations take the simple form

$$B(x, y)B(y, z) = B(x, z), \quad B(x, y) = B(y, x)^{-1} \tag{36}$$

for almost all $(x, y, z) \in \mathbb{R}^3$ (respectively, for almost all $(x, y) \in \mathbb{R}^2$) .

Remark 3. If we knew that (36) is true not only almost everywhere, but everywhere, this equation would be easy to solve. Indeed, put $C(x) = B(0, x) = B^*(x, 0)$. Then (36) for $z = 0$ implies that $B(x, y) = C(x^{-1})C(y)$. Conversely, for any unitary-valued operator function $C(x)$ the expression $B(x, y) = C(x)^{-1}C(y)$ satisfies (36). ♡

Lemma 4. *Any measurable unitary-valued operator function $B(x, y)$, which satisfies* (36) *for almost all x, y, z, has the form*

$$B(x, y) = C(x)C(y)^{-1} \quad \textit{for almost all } (x, y) \in \mathbb{R}^2 \tag{37}$$

where $C(x)$ is a measurable unitary-valued operator function on $\mathbb{R}$.

From the lemma we get $u(s) = A(s, x) \otimes U(s) = C(x)C(x+s)^{-1} \otimes U(s)$ and the transformation $f(x) \mapsto C(x)f(x)$ sends $u(s)$ to the desired form $1 \otimes U(s)$ and preserves the form of the operators $v(t) = 1 \otimes V(s)$.

Hence, our realization is indeed a direct sum of n copies of the standard one.

Proof of Lemma 4. To make the exposition easier, we consider at first the case $\dim W = 1$.

Then $B(x, y)$ will be simply a complex-valued function with absolute value 1 almost everywhere. We shall denote it $b(x, y)$. Our goal is to find a complex-valued function $c(x)$ of absolute value 1 such that $b(x, y) = c(x)^{-1}c(y)$ almost everywhere.

Let α, β, γ be test functions from $\mathcal{A}(\mathbb{R})$. Introduce the notation

$$f_\alpha(y) := \int_{\mathbb{R}} \alpha(x)b(x, y)dx, \quad c_{\alpha,\gamma} := \int_{\mathbb{R}^2} \alpha(x)b(x, z)\overline{\gamma(z)}dxdz.$$

Let us multiply the basic relation $b(x, y)b(y, z) = b(x, z)$ by $\alpha(x)\cdot\beta(y)\cdot\overline{\gamma(z)}$ and integrate over $\mathbb{R}^3$. Using the relation $b(y, z) = \overline{b(z, y)}$, the result can be written as

$$\int_{\mathbb{R}} f_\alpha(y)\overline{f_\gamma(y)}\beta(y)dy = c_{\alpha,\gamma}\int_{\mathbb{R}} \beta(y)dy.$$

It follows that $f_\alpha(y)\overline{f_\gamma(y)} = c_{\alpha,\gamma}$ as a distribution on $\mathbb{R}$, or

$$f_\alpha(y)\overline{f_\gamma(y)} = c_{\alpha,\gamma} \quad \text{almost everywhere.} \tag{38}$$

From (38) we conclude that $c_{\alpha,\alpha} = |f_\alpha(y)|^2 \geq 0$ and $f_\alpha(y) = \frac{c_{\alpha,\gamma}}{c_{\gamma,\gamma}}f_\gamma(y)$ if $c_{\gamma,\gamma} \neq 0$.

Hence, all functions $f_\alpha(x)$, $\alpha \in \mathcal{A}(\mathbb{R})$, are proportional to some fixed function with absolute value 1, which we denote by $c(x)$. More precisely, we have

$$f_\alpha(x) = c_\alpha \cdot c(x), \qquad \text{where } |c_\alpha|^2 = c_{\alpha,\alpha} \text{ and } c_\alpha\overline{c_\gamma} = c_{\alpha,\gamma}.$$

Put $\langle\alpha\rangle = \int_{\mathbb{R}} \alpha(x)\overline{c(x)}dx$. Then from equalities

$$\int_{\mathbb{R}^2} \alpha(x)b(x, z)\overline{\gamma(z)}dxdz = \int_{\mathbb{R}} f_\alpha(z)\overline{\gamma(z)}dz = \int_{\mathbb{R}} \alpha(x)\overline{f_\gamma(x)}dx,$$

we get $c_{\alpha,\gamma} = c_\alpha \cdot \overline{\langle\gamma\rangle} = \langle\alpha\rangle \cdot \overline{c_\gamma}$, which implies $\langle\alpha\rangle = c_\alpha$. It follows that $b(x, z) = \overline{c(x)}c(y)$ as desired.

The general case $\dim W > 1$ is technically more difficult, but the scheme is the same. The operator function $B(x, y)$ satisfies the relations

$$B(x, y)B(y, z) = B(x, z), \qquad B(x, z) = B^*(z, x) = B^{-1}(z, x). \tag{39}$$

Let α, γ be test vector-functions on $\mathbb{R}$ with values in W. We introduce the vector-valued functions $F_\gamma(y)$ and the numbers $c_{\alpha,\gamma}$ by

$$F_\gamma(y) = \int_{\mathbb{R}} B(y, z)\gamma(z)dz, \qquad c_{\alpha,\gamma} = \int_{\mathbb{R}^2} \big(B(x, y)\alpha(y), \gamma(x)\big)_W dxdy.$$

Then from the relations (39) we get for any $\beta \in \mathcal{A}(\mathbb{R})$:

$$\int_{\mathbb{R}} \big(F_\alpha(y),\, F_\gamma(y)\big)_W \beta(y)dy = c_{\alpha,\gamma} \int_{\mathbb{R}} \beta(y)dy.$$

This implies that $\big(F_\alpha(y),\, F_\gamma(y)\big)_W = c_{\alpha,\gamma}$ for almost all $y \in \mathbb{R}$.

Geometrically, the last relation means that the configuration of vectors $\{F_\alpha(y), \alpha \in \mathcal{A}(\mathbb{R})\}$ has the same shape for almost all $y \in \mathbb{R}$. Therefore, there exists a collection of vectors $\{v_\alpha,\, \alpha \in \mathcal{A}(\mathbb{R})\}$ and a unitary operator $C(y)$ such that $F_\alpha(y) = C(y)v_\alpha$ for almost all $y \in \mathbb{R}$ and all $\alpha \in \mathcal{A}(\mathbb{R})$. It follows that $B(y,\, z) = C(y)C^{-1}(z)$ almost everywhere. □

The third proof is based on the Inducibility Criterion from Appendix V.2.1. Namely, consider the representation $(\pi,\, H)$ of the Heisenberg group defined by the formula

$$\pi(a,\, b,\, c) = e^{i\hbar c} v(b)u(a). \tag{40}$$

It is easy to check, using the Weyl form of CCR, that (40) is indeed a unitary representation of the Heisenberg group (see Section 3.1).

Repeating the first part of the previous proof, we can assume that π acts in $L^2(\mathbb{R},\, W,\, dx)$ by the formula

$$(\pi(a,\, b,\, c)f)(x) = e^{i(\hbar c + bx)} A(a,\, x) f(x + \hbar a). \tag{41}$$

We also define the $*$-representation $(\Pi,\, H)$ of $\mathcal{A}(\mathbb{R})$ by

$$(\Pi(\phi)f)(x) = \phi(x)f(x). \tag{42}$$

From (41) and (42) we conclude that the representation (40) of the Heisenberg group is compatible with the representation $(\Pi,\, H)$ of $\mathcal{A}(\mathbb{R})$ if we consider $\mathbb{R}$ as a homogeneous manifold where the Heisenberg group acts by the rule

$$(a,\, b,\, c) \cdot x = x + \hbar a. \tag{43}$$

According to the Inducibility Criterion, $(\pi,\, H)$ is induced from some representation $(\rho,\, W)$ of the stabilizer of some point $x_0 \in \mathbb{R}$. If $(\pi,\, H)$ is irreducible, so is $(\rho,\, W)$. But the group $Stab(x_0)$ is abelian, so its irreducible representations are 1-dimensional and have the form $(0,\, b,\, c) \mapsto e^{\hbar c + \lambda b}$.

The induced representation, constructed along the standard procedure described in Appendix V.2.2, looks like

$$(\pi(a,\, b,\, c)f)(x) = e^{i(\hbar c + b(x+\lambda))} f(x + \hbar a). \tag{44}$$

Exercise 11. Show that representations (44) for any two values of $\lambda \in \mathbb{R}$ are equivalent.

Hint. Use the shift operator to intertwine both unirreps.

It remains to observe that for $\lambda = 0$ the formula (44) coincides with the standard representation. □

3. Representation theory for the Heisenberg group

Here we collect the main facts about the unirreps of H using the results of the previous sections.

3.1. The unitary dual $\widehat{H}$.

We recall that by the unitary dual $\widehat{G}$ we mean the set of equivalence classes of all unirreps for the given topological group G. This set has a natural topology (see Appendix V.2 and Chapter 3) and also can be viewed as a "non-commutative manifold", which we discuss later.

There is a straightforward connection between CCR in the Weyl form and unitary representations of the Heisenberg group. Indeed, let us write the general element $g_{a,b,c}$ of H in the form

$$g_{a,b,c} = \exp cZ \exp bY \exp aX = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

The 1-parametric subgroups of H satisfy

$$\exp aX \exp bY = \exp abZ \exp bY \exp aX.$$

Therefore, if $u(s)$, $v(t)$ satisfy (23), we can define for any $\lambda \in \mathbb{R}$ the unitary representation π_λ of H by the formula

$$\pi_\lambda(g_{a,b,c}) = e^{2\pi i\lambda c} v(\lambda b) u\left(\frac{a}{\hbar}\right).$$

This representation is irreducible, iff the representation $\{u(s), v(t)\}$ of CCR has that property. In particular, for the standard representation of CCR we have

$$[\pi_\lambda(g_{a,b,c})f](x) = e^{2\pi i\lambda(c+bt)} f(x+a). \tag{45}$$

Conversely, if π is any unitary irreducible representation of H, then the elements of the center are represented by scalar operators. Hence, we have

$$\pi(\exp cZ) = e^{2\pi i\lambda c} \cdot 1 \qquad \text{for some } \lambda \in \mathbb{R}. \tag{46}$$

If $\lambda \neq 0$, we can define the irreducible representation of CCR in the Weyl form by

$$u(s) = \pi(\exp s\hbar X), \qquad v(t) = \pi(\exp t\lambda^{-1} Y). \tag{47}$$

It is clear that the two correspondences are reciprocal. So, due to the uniqueness theorem, for any real number $\lambda \neq 0$ there exists exactly one

unirrep π of the Heisenberg group satisfying (46), namely the representation π_λ given by (45).

The remaining irreducible representations of H are trivial on the center $C = \exp \mathbb{R}\cdot Z$. Hence, they are actually representations of the abelian group H/C and must be 1-dimensional. The general form of such representations is

$$\pi_{\mu,\nu}(g_{a,b,c}) = e^{2\pi i(a\mu+b\nu)}. \tag{48}$$

The final result can be formulated as

Theorem 5. *The unitary dual $\widehat{H}$ for the Heisenberg group H splits into two parts:*

a) *the* 1*-parametric family of equivalence classes of infinite-dimensional unirreps π_λ, $\lambda \neq 0$, and*

b) *the* 2*-parametric family of* 1*-dimensional representations $\pi_{\mu,\nu}$.* □

Note that the equivalence class of π_λ can be realized as a unirrep in the space $\mathcal{H}$ of holomorphic functions on $\mathbb{C}$. Here the representation operators are given by

$$\big(\pi_\lambda(a,\, b,\, c)F\big)(z) = e^{\alpha z+\beta}F(z-\overline{\alpha}) \tag{49}$$

where $\alpha = \frac{b+i\lambda a}{\sqrt{2}}$, $\beta = -\frac{\alpha\overline{\alpha}}{2} + i\lambda(\frac{ab}{2}+c)$.

We leave to the reader to write explicitly the third realization of the same equivalence class in the space of sections of a line bundle over the 2-torus.

Thus, $\widehat{H} = \mathbb{R}\backslash\{0\} \cup \mathbb{R}^2$ as a set. It is reasonable to expect that the topology of $\widehat{H}$ is the natural one on each of the pieces $\mathbb{R}\backslash\{0\}$ and $\mathbb{R}^2$. A much more interesting question is: what is the closure of $\mathbb{R}\backslash\{0\}$ in $\widehat{H}$? Or, in other words, what is the limit[4] $\lim_{\lambda\to 0}\pi_\lambda$?

If we simply put $\lambda = 0$ in (45), we obtain a highly reducible representation which actually is a continuous sum of $\pi_{\mu,0}$, $\mu \in \mathbb{R}$. However, it does not yet mean that the limit contains only representations with $\nu = 0$.

For example, if we make a Fourier transform in (45) and only after that put $\lambda = 0$, we get another reducible representation, namely, the continuous sum of $\pi_{0,\nu}$, $\nu \in \mathbb{R}$. And if we make an appropriate transform in (49) and then put $\lambda = 0$, we get the continuous sum of all 1-dimensional unirreps.

It turns out that the last answer is correct: the limit of π_λ when $\lambda \to 0$ is the set of all 1-dimensional unirreps. We omit the (rather easy) proof of it. A more general fact can be found in Chapter 3.

[4]Since the topological space $\widehat{H}$ is not Hausdorff, the limit of a sequence can be more than one point.

3.2. The generalized characters of H.

The notion of a generalized character is explained in Appendix V.1.4.

For 1-dimensional representations $\pi_{\mu,\nu}$ the generalized characters coincide with multiplicative characters and with representations themselves.

For the unirreps π_λ generalized characters are non-regular generalized functions which can be computed explicitly. To do this, we pick a $\phi \in \mathcal{A}(H)$. The operator

$$\pi_\lambda(\phi) := \int_H \phi(a,\, b,\, c)\pi_\lambda\big(g(a,\, b,\, c)\big)dadbdc$$

is an integral operator in $L^2(\mathbb{R},\, dx)$ with the kernel

$$K_\phi(x,\, y) = \int_{\mathbb{R}^2} \phi(y - x,\, b,\, c)e^{2\pi i\lambda(bx+c)}dbdc.$$

This kernel is a rapidly decreasing function on $\mathbb{R}^2$. Therefore, the operator $\pi_\lambda(\phi)$ is of trace class and its trace is given by the formula

$$\begin{aligned} \operatorname{tr}\pi_\lambda(\phi) &= \int_{\mathbb{R}} K_\phi(x,\, x)dx = \int_{\mathbb{R}^3} \phi(0,\, b,\, c)e^{2\pi i\lambda(bx+c)}dbdcdx \\ &= \frac{1}{\lambda}\int_{\mathbb{R}} \phi(0,\, 0,\, c)e^{2\pi i\lambda c}dc. \end{aligned}$$

So, as a distribution,

$$\chi_\lambda(a,\, b,\, c) = \frac{1}{\lambda}\delta(a)\delta(b)e^{2\pi i\lambda c}. \tag{50}$$

Introduce the Fourier transform

$$\widetilde{\phi}(x,\, y,\, z) := \int_{\mathbb{R}^3} \phi(a,\, b,\, c)e^{2\pi i(ax+by+cz)}dxdydz.$$

In terms of $\widetilde{\phi}$ the trace of $\pi_\lambda(\phi)$ can be written very simply:

$$\operatorname{tr}\pi_\lambda(\phi) = \int_{\mathbb{R}^2} \widetilde{\phi}(x,\, y,\, \lambda)\,\frac{dx \wedge dy}{\lambda}. \tag{51}$$

We have used above the canonical coordinates on the group H. However, the final formula is also valid in exponential coordinates. These coordinates identify the Lie group H with its Lie algebra $\mathfrak{h}$. Therefore, the Fourier transform $\widetilde{\phi}$ lives naturally in the dual space $\mathfrak{h}^*$ with coordinates $x,\, y,\, z$.

We see that the generalized character is the Fourier transform of the measure $\frac{dx\wedge dy}{z}$, concentrated on the hyperplane $z = \lambda$.

3.3. The infinitesimal characters of H.

The infinitesimal character I_π of a unirrep π of H is defined in Appendix V.1.3 as a homomorphism of the center $Z(\mathfrak{h})$ of the enveloping algebra $U(\mathfrak{h})$ into the complex field $\mathbb{C}$, given by

$$\pi(A) = I_\pi(A) \cdot 1 \qquad \text{for all } A \in Z(\mathfrak{h}).$$

As we have seen in Section 2.2, the center of $U(\mathfrak{h})$ is just $\mathbb{C}[Z]$. So, the infinitesimal character I_π of a unirrep π is defined by one number $I_\pi(Z)$.

From the formulae (45) and (48) we immediately obtain

$$I_{\pi_\lambda}(Z) = 2\pi i\lambda, \qquad I_{\pi_{\mu,\nu}}(Z) = 0. \tag{52}$$

Thus, the infinite-dimensional unirreps are determined by their infinitesimal characters up to equivalence, while all 1-dimensional unirreps have the same infinitesimal character.

Later we shall see that this situation is typical: the infinitesimal characters form a coordinate system on the set of "generic" representations and do not separate the "degenerate" ones.

3.4. The tensor product of unirreps.

Using the explicit formulae (45) and (48) it is easy to decompose the product of two unirreps into irreducible components. We consider here the case of two infinite-dimensional unirreps π_{λ_1}, π_{λ_2}. Their tensor product acts in the Hilbert space $L^2(\mathbb{R}, |dx|) \boxtimes L^2(\mathbb{R}, |dy|) \simeq L^2(\mathbb{R}^2, |dx \wedge dy|)$ by the formula

$$\big((\pi_{\lambda_1} \otimes \pi_{\lambda_2})(a, b, c)F\big)(x, y) = e^{2\pi i\big(\lambda_1(bx+c)+\lambda_2(by+c)\big)} F(x+a, y+a).$$

Denote by L_s the line given by the equation $x - y = s$ and by H_s the Hilbert space of square-integrable functions on L_s. We see that the representation above is a continuous sum of representations $\widetilde{\pi}_s$ acting by the same formula in the spaces H_s.

We choose $t = \frac{\lambda_1 x + \lambda_2 y}{\lambda_1 + \lambda_2}$ as a parameter on the line L_s. Then the representation $\widetilde{\pi}_s$ takes the form

$$\big(\widetilde{\pi}_s(a, b, c)\phi\big)(t) = e^{2\pi i(\lambda_1+\lambda_2)(bt+c))}\phi(t+a).$$

It follows that all $\widetilde{\pi}_s$ are equivalent to $\pi_{\lambda_1+\lambda_2}$, provided that $\lambda_1 + \lambda_2 \neq 0$. Thus,

$$\pi_{\lambda_1} \otimes \pi_{\lambda_2} = \infty \cdot \pi_{\lambda_1+\lambda_2}. \tag{53}$$

This result is in accordance with Rule 5 of the User's Guide (see the Introduction) and with the count of functional dimensions. Indeed, the functional dimension of π_λ is 1, so the product $\pi_{\lambda_1} \otimes \pi_{\lambda_2}$ has the functional dimension 2, as well as the right-hand side of (53). Note, in this connection, the general formula

$$\bigotimes_{k=1}^{n} \pi_{\lambda_k} = \infty^{n-1} \cdot \pi_\lambda, \quad \text{provided} \quad 0 \neq \lambda := \sum_{k=1}^{n} \lambda_k.$$

Exercise 12. Prove the relations

$$\begin{gathered} \pi_\lambda \otimes \pi_{-\lambda} = \int \pi_{\mu,\nu} |d\mu \wedge d\nu|, \quad \pi_\lambda \otimes \pi_{\mu,\nu} = \pi_\lambda, \\ \pi_{\mu_1,\nu_1} \otimes \pi_{\mu_2,\nu_2,} = \pi_{\mu_1+\mu_2,\nu_1+\nu_2}. \end{gathered} \tag{54}$$

♣

4. Coadjoint orbits of the Heisenberg group

4.1. Description of coadjoint orbits.

We shall use the matrix realization (2) for the Lie algebra $\mathfrak{h}$. The dual space $\mathfrak{g}^*$ is identified with the space of lower-triangular matrices of the form

$$F = \begin{pmatrix} * & * & * \\ x & * & * \\ z & y & * \end{pmatrix}.$$

Here the stars remind us that we actually consider the quotient space of $\mathrm{Mat}_3(\mathbb{R})$ by the subspace $\mathfrak{g}^\perp$ of upper-triangular matrices (including the diagonal).

The group H consists of upper-triangular matrices

$$g = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

The coadjoint action is

(55)

$$\begin{aligned} K(g)F = p(gFg^{-1}) &= \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} * & * & * \\ x & * & * \\ z & y & * \end{pmatrix} \cdot \begin{pmatrix} 1 & -a & -c+ab \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} * & * & * \\ x+bz & * & * \\ z & y-az & * \end{pmatrix} \end{aligned}$$

or

$$K\left(g_{a,b,c}\right)(x,\, y,\, z) \;=\; (x+bz,\, y-az,\, z).$$

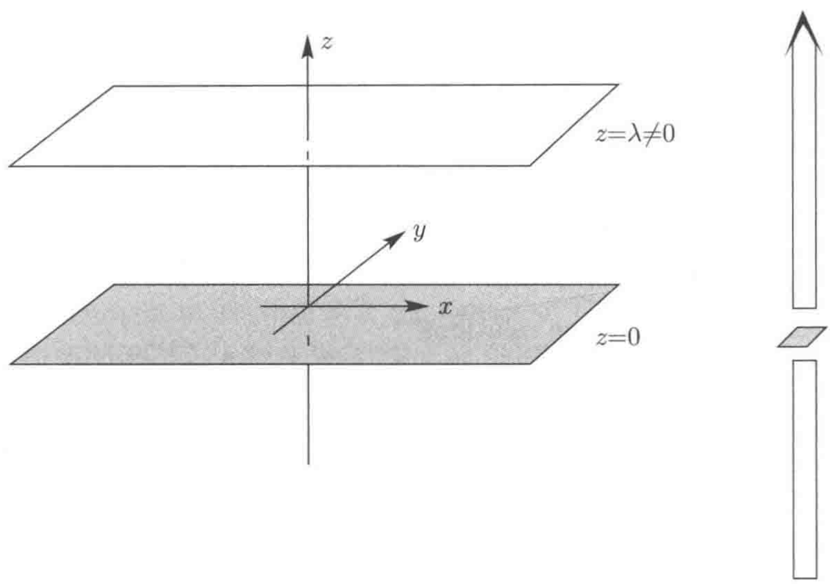

Figure 1

We come to the following result.

Theorem 6. *The set $\mathcal{O}(H)$ of all coadjoint orbits for the Heisenberg group H consists of*

a) *the 2-dimensional planes Ω_λ given by the equation $z = \lambda \neq 0$ and*

b) *the points $\Omega_{\mu,\nu}$ given by the equations $x = \mu$, $y = \nu$, $z = 0$.* □

The picture of $\mathcal{O}(H)$ is shown in Figure 1.

4.2. Symplectic forms on orbits and the Poisson structure on $\mathfrak{h}^*$.

Let us compute the symplectic structure on the orbit Ω_λ. The coadjoint action (55) associates to the basic vectors X, Y, Z the vector fields $K_*(X)$, $K_*(Y)$, $K_*(Z)$ on $\mathfrak{h}^*$ which are tangent to Ω_λ:

$$K_*(X) = -z\partial_y, \qquad K_*(Y) = z\partial_x, \qquad K_*(Z) = 0. \tag{56}$$

The symplectic form σ is defined by

$$\sigma(F)(K_*(X), K_*(Y)) = \langle F, [X, Y]\rangle \quad \text{or} \quad \sigma(x, y, z)(-z\partial_y, z\partial_x) = z.$$

Therefore, we get

$$\sigma = \frac{1}{z}\, dx \wedge dy. \tag{57}$$

It follows that

$$\text{s-grad}\, x = -z\partial_y = K_*(X), \ \text{s-grad}\, y = z\partial_x = K_*(Y), \ \text{s-grad}\, z = 0 = K_*(Z),$$

in agreement with the general statement

$$\text{(58)} \qquad \text{s-grad}\, f_X = K_*(X) \qquad \text{where } f_X(F) = \langle F,\, X \rangle.$$

The canonical Poisson structure on $\mathfrak{h}^*$ in coordinates x, y, z is given by

$$\{f_1,\, f_2\} = z\left(\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}\right).$$

We see that the symplectic leaves are the planes $z = \mathit{const} \neq 0$ and the points $(x,\, y,\, 0)$ of the plane $z = 0$, i.e. precisely the coadjoint orbits.

4.3. Projections of coadjoint orbits.

Let $A \subset H$ be a closed subgroup, and let $\mathfrak{a} = \mathrm{Lie}\,(A)$ be its Lie algebra. According to the ideology of the orbit method, the functors Ind_A^H and Res_A^H are related to the canonical projection $p_{\mathfrak{a}} : \mathfrak{g}^* \to \mathfrak{a}^*$. So, we consider here the geometry of these projections for different subalgebras $\mathfrak{a}$.

First let $\dim \mathfrak{a} = 1$. There are two different cases.

1. $\mathfrak{a} = \mathfrak{c} = \mathbb{R} \cdot Z$ is the central subalgebra. Looking at Figure 1, the projection $p_{\mathfrak{a}}$ is just the horizontal projection to the vertical axis. Therefore, for any orbit $\Omega \subset \mathfrak{g}^*$ the image $p(\Omega)$ is just one point:

$$p_{\mathfrak{a}}(\Omega_\lambda) = \lambda, \qquad p_{\mathfrak{a}}(\Omega_{\mu,\nu}) = 0.$$

2. $\mathfrak{a} = \mathbb{R} \cdot (\alpha X + \beta Y + \gamma Z)$ is a non-central subalgebra. Then, in the same picture, $p_{\mathfrak{a}}$ is a non-horizontal projection to some non-vertical line. The image of Ω_λ is the whole line, and the image of $\Omega_{\mu,\nu}$ is the point $\alpha\mu + \beta\nu$.

Now consider a 2-dimensional subalgebra $\mathfrak{a}$. It can be written as $F^\perp$ where $F \in \mathfrak{g}^*$ has the property $\langle F,\, Z \rangle = 0$.

Looking again at Figure 1, we can view $p_{\mathfrak{a}}$ as the projection along the horizontal line $\mathbb{R} \cdot F$ to an orthogonal 2-dimensional plane. Therefore, $p_{\mathfrak{a}}(\Omega_\lambda)$ is a line in $\mathfrak{a}^*$, while $p_{\mathfrak{a}}(\Omega_{\mu,\nu})$ is a point.

5. Orbits and representations

In this section we compare the set $\mathcal{O}(H)$ of coadjoint orbits with the set $\widehat{H}$ of equivalence classes of unirreps. According to Theorems 4 and 5, both are the union of two pieces: the real line with the origin deleted and the real plane.

It suggests the following one-to-one correspondence between orbits and (equivalence classes of) unirreps:

$$\Omega_\lambda \iff \pi_\lambda, \qquad \Omega_{\mu,\nu} \iff \pi_{\mu,\nu}.$$

We show below that this correspondence allows us to formulate the main results of the representation theory of H in simple geometric terms. Moreover, these formulations can be extended to wide classes of Lie groups, as we show in subsequent chapters.

5.1. Restriction-induction principle and construction of unirreps.

We start with the following general statement which is an essential part of the ideology of the orbit method.

Restriction-induction principle:

If $H \subset G$ is a closed subgroup and $p : \mathfrak{g}^* \to \mathfrak{h}^*$ is the natural projection (the restriction to $\mathfrak{h}$ of a linear functional on $\mathfrak{g}$), then the restriction and induction functors can be naturally described in terms of the projection p.

In the next chapter we make this statement more precise and write it in the form

$$\text{a) } \operatorname{Res}_H^G \pi_\Omega = \int_{\omega \subset p(\Omega)} m(\Omega, \omega) \pi_\omega d\omega,$$

$$\text{b) } \operatorname{Ind}_H^G \pi_\omega = \int_{\Omega \subset K(G) p^{-1}(\omega)} m(\Omega, \omega) \pi_\Omega d\Omega$$

where

$\pi_\Omega \in \widehat{G}$ is the unirrep of G associated to an orbit $\Omega \in \mathcal{O}(G)$;

$\pi_\omega \in \widehat{H}$ is the unirrep of H associated to an orbit $\omega \in \mathcal{O}(H)$;

$\operatorname{Res}_H^G \pi_\Omega$ is the restriction of π_Ω on H;

$\operatorname{Ind}_H^G \pi_\omega$ is the representation of G induced by π_ω;

$m(\Omega, \omega)$ is the multiplicity factor, which here takes only values $0, 1, \infty$;

integrals denote the continuous sum of representations (see Appendix IV.2.5) with respect to appropriate measures $d\omega$ and $d\Omega$ on the sets of orbits.

Right now we prefer to use the restriction-induction principle in a somewhat fuzzy form in combination with common sense.

We start with the simplest case of an abelian Lie group, $G = \mathbb{R}^n$. The Lie algebra $\operatorname{Lie}(G)$ is the vector space $\mathbb{R}^n$ with the zero commutator. All coadjoint orbits are single points in the dual vector space $(\mathbb{R}^n)^*$.

Compare it with the set of unirreps for $\mathbb{R}^n$. Recall that for any abelian group A all unirreps are 1-dimensional and are just multiplicative characters.

Let A be a locally compact abelian topological group. The set of all continuous multiplicative characters of A is denoted by $\widehat{A}$. It is endowed with the group structure (ordinary multiplication of functions) and with

the topology of uniform convergence on compact sets. It turns out that $\widehat{A}$ is also an abelian locally compact topological group which is called the **Pontryagin dual** of A.

Pontryagin duality principle. a) *The canonical map* $\Phi : A \to \hat{\hat{A}}$, *given by*

$$\big(\Phi(a)\big)(\chi) = \chi(a),$$

is an isomorphism of topological groups.

b) *For any invariant measure da on A there exists a dual invariant measure $d\chi$ on $\hat{A}$ such that the direct and inverse Fourier transforms*

$$\widehat{F} : f(a) \rightsquigarrow \widehat{f}(\chi) = \int_A f(a)\overline{\chi(a)}da \quad \textit{and} \quad \check{F} : \phi(\chi) \rightsquigarrow \check{\phi}(\chi) =: \int_{\widehat{A}} \phi(\chi)\chi(a)d\chi$$

are reciprocal unitary isomorphisms between $L^2(A, da)$ and $L^2(\widehat{A}, d\chi)$. □

In particular, the Pontryagin dual $\widehat{\mathbb{R}^n}$ to the abelian group $\mathbb{R}^n$ is itself isomorphic to $\mathbb{R}^n$ and can be identified with the dual vector space $\big(\mathbb{R}^n\big)^*$. We choose the identification which associates to $\lambda \in \big(\mathbb{R}^n\big)^*$ the character $\chi_\lambda(x) = e^{2\pi i\langle\lambda,\, x\rangle}$. The advantage of this choice is that the standard Lebesgue measures dx and $d\lambda$ are dual to each other (there are no additional factors in the direct and inverse Fourier transforms).

Now we consider the one-point coadjoint orbits for any simply connected Lie group G and relate them to 1-dimensional representations of G. Let $\{F\}$ be a one-point orbit in $\mathfrak{g}^*$, and let $\pi_{\{F\}}$ be the corresponding representation of G.

The restriction of this representation to a 1-parametric subgroup $\exp \mathbb{R} \cdot X$ must correspond to the restriction of F to the subalgebra $\mathbb{R} \cdot X$. Using the identification $\lambda \longleftrightarrow \chi_\lambda$ chosen above, we conclude that the representation $\pi_{\{F\}}$ must have the form

$$\pi_{\{F\}}(\exp X) = e^{2\pi i\langle F,\, X\rangle}. \tag{59}$$

Remark 4. Formula (59) indeed defines a 1-dimensional unirrep of G. The point is that F is a fixed point for the coadjoint action. Therefore, $stab(F) = \mathfrak{g}$, which is equivalent to the condition $F\,\big|_{[\mathfrak{g},\mathfrak{g}]} = 0$. Hence, F defines a linear functional $\widetilde{F}$ on the abelian Lie algebra $\widetilde{\mathfrak{g}} := \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$.

Let $\widetilde{G}$ be the simply connected Lie group corresponding to the Lie algebra $\widetilde{\mathfrak{g}}$ (as sets both $\widetilde{G}$ and $\widetilde{\mathfrak{g}}$ can be identified with $\mathbb{R}^d$, $d = \dim \widetilde{\mathfrak{g}}$). The initial group G factors through the Lie group $\widetilde{G} = G/[G,G]$ and (59) actually means

$$\pi_{\{F\}}(\exp X) = \widetilde{\pi_{\{F\}}}\big(p(\exp X)\big) = e^{2\pi i\langle\widetilde{F},\, p_*(X)\rangle}$$

where $p: G \to G/[G,G]$ and $p_*: \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ are the canonical projections. The last expression is clearly a character of G. ♡

Coming back to the Heisenberg group, we see that a one-point orbit $\Omega_{\mu,\nu}$ must correspond exactly to the 1-dimensional unirrep $\pi_{\mu,\nu}$.

Now consider a 2-dimensional orbit Ω_λ. We want to write the associated unirrep in the form $\mathrm{Ind}_A^H \rho$ where A is a closed subgroup of H and ρ is a 1-dimensional unirrep of A.

Let $F \in \mathfrak{a}^*$ be the functional on $\mathfrak{a}$ associated to ρ. Then, according to the restriction-induction principle, $\mathrm{Ind}_A^H \rho$ splits into unirreps π_Ω for which Ω intersects $p^{-1}(F)$. There are two cases when $p^{-1}(F)$ is contained in a single orbit Ω_λ:

a) $\mathfrak{a}$ is the central subalgebra $\mathfrak{c} = \mathbb{R} \cdot Z$ and

$$\langle F,\, Z\rangle = \lambda; \tag{60}$$

b) $\mathfrak{a}$ is any 2-dimensional subalgebra[5] and $F \in \mathfrak{a}^*$ satisfies (60).

In case a) the induced representation has the functional dimension 2 (i.e. is naturally realized in the space of functions of two variables). Hence, it is too big to be irreducible. We shall see later that it splits into a countable set of unirreps equivalent to π_λ.

Consider the case b). Let us choose $\mathfrak{a} = \mathbb{R} \cdot Y \oplus \mathbb{R} \cdot Z$ and define $F \in \mathfrak{a}^*$ by $\langle F,\, \beta Y + \gamma Z\rangle = \lambda\gamma$. Then the inducing representation of A has the form

$$\rho(\exp(\beta Y + \gamma Z)) = e^{2\pi i \lambda \gamma}.$$

Let us derive the explicit formula for $\mathrm{Ind}_A^H \rho$. We identify the homogeneous manifold $M = H/A$ with $\mathbb{R}$ and define the section $s: M \to H$ by

$$s(x) = \exp xX.$$

The master equation (see Appendices V.2.1 and V.2.2) takes the form

$$\exp(xX) \cdot g_{a,b,c} = \exp(\beta Y + \gamma Z) \cdot \exp(yX)$$

or

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with given x, a, b, c and unknown β, γ, y. The solution is

$$\beta = b, \qquad \gamma = c + bx, \qquad y = x + a.$$

[5] Recall that $\mathfrak{a}$ is necessarily abelian and contains $\mathfrak{c}$ (see Exercise 2 in Section 1.1).

Therefore, the induced representation $\mathrm{Ind}_A^H \rho$ is given by

$$\left(\mathrm{Ind}_A^H \rho(g_{a,b,c}) f\right)(x) = e^{2\pi i \lambda (c+bx)} f(x+a),$$

and coincides with π_λ given by (45).

Now we shall show that the equivalence class of the induced representation $\mathrm{Ind}_A^H \rho$ does not change if we choose another subalgebra $\mathfrak{a}' \subset \mathfrak{h}$ or another unirrep ρ' associated with a different linear functional $F' \in \mathfrak{a}'^*$ (subjected as before to the condition $\langle F', Z\rangle = \lambda$).

Actually, these two possibilities are related.

Lemma 5. *Let $\mathfrak{a}' = g \cdot \mathfrak{a} \cdot g^{-1}$ and $F' = K(g)F \in \mathfrak{a}'^*$. Let ρ' be the 1-dimensional unirrep of $A' = \exp \mathfrak{a}'$ given by $\rho'(\exp X) = e^{2\pi i \langle F', X\rangle}$. Then*

$$\mathrm{Ind}_{A'}^H \rho' \sim \mathrm{Ind}_A^H \rho.$$

Proof. Let $M = H/A$ and $M' = H/A'$. We identify M and M' with $\mathbb{R}$ as before and choose in the second case the section $s' : M' \to H$ by

$$s'(x) = g \cdot \exp xX \cdot g^{-1}.$$

Then the master equation in the second case can be obtained from the initial master equation via conjugation by g. Therefore, the induced representation $\mathrm{Ind}_{A'}^H \rho'$ is related to $\mathrm{Ind}_A^H \rho$ by the formula

$$\mathrm{Ind}_{A'}^H \rho'(g \cdot g_{a,b,c} \cdot g^{-1}) = \mathrm{Ind}_A^H \rho(g_{a,b,c}).$$

$\square$

Lemma 5 implies that often it is enough to vary *either* $\mathfrak{a}$ *or* F. In particular, in our case we can assume that $\langle F, X\rangle = \langle F, Y\rangle = 0$, $\langle F, Z\rangle = \lambda$ and $\mathfrak{a}$ is an arbitrary 2-dimensional subalgebra of $\mathfrak{h}$ which contains the center $\mathfrak{z}$. (Note that this subalgebra is actually an ideal in $\mathfrak{h}$, hence does not vary under inner automorphisms.)

There are two different cases:

1. $\mathfrak{a}$ does not contain X; then we can identify G/A with $\mathbb{R}$ using the section $s(x) = \exp xX$.

2. $\mathfrak{a} = \mathbb{R} \cdot X \oplus \mathbb{R} \cdot Z$; then we can identify G/A with $\mathbb{R}$ using the section $s(x) = \exp xY$.

We denote the induced representations of H in these two cases by π_1 and π_2, respectively. It is easy to compute the corresponding representations of

the Lie algebra $\mathfrak{h}$. The result is:

$$\begin{array}{cccc} & X & Y & Z \\ (\pi_1)_* & \frac{d}{dx} & i\lambda x + \mu & i\lambda \\ (\pi_2)_* & -i\lambda x + \nu & \frac{d}{dx} & i\lambda \end{array}$$

We see that both π_1 and π_2 are equivalent to π_λ. In the first case the intertwiner is the shift operator $T_{\mu/\lambda} : f(x) \mapsto f(x + \mu/\lambda)$. In the second case it is the shift operator $T_{-\nu/\lambda}$ composed with the Fourier transform.

5.2. Other rules of the User's Guide.

The results of the previous section can be formulated as follows: the correspondence

$$\Omega_\lambda \leftrightarrow \pi_\lambda, \qquad \Omega_{\mu,\nu} \leftrightarrow \pi_{\mu,\nu} \tag{61}$$

is forced by the ideology of the orbit method with respect to the restriction-induction functor. It turns out that conversely, if we fix the correspondence (61), then all rules of the User's Guide will be correct. We leave this to the reader to check using the results described above in this chapter.

6. Polarizations

The general results about invariant polarizations look especially transparent in the case of the coadjoint orbits of the Heisenberg groups. We give here the complete description of real and complex invariant polarizations.

6.1. Real polarizations.

Consider in more detail the symplectic geometry of coadjoint orbits. We leave aside the trivial case of one-point orbits and consider a 2-dimensional orbit Ω_λ, $\lambda \in \mathbb{R}\backslash\{0\}$. It is a 2-dimensional plane which is a homogeneous symplectic manifold with respect to the group $K(H)$ acting by translations.

We are interested in the real polarizations of Ω_λ that are invariant under translation. It is clear that every such polarization is just a splitting of Ω_λ into the union of parallel lines $ax + by = const.$

In accordance with the general theory (see Chapter 1), these lines are just the orbits of the subgroup $K(A) \subset K(H)$ where $A \subset H$ is a 2-dimensional subgroup of H. We observe that in this case all 2-dimensional subalgebras $\mathfrak{a} \subset \mathfrak{h}$ are abelian, hence are subordinate to any functional.

The functions on Ω_λ which are constant along the leaves of the polarization have the form $F(x, y) = f(ax + by)$ and form an abelian subalgebra in $C^\infty(\Omega_\lambda)$ with respect to the Poisson bracket.

6.2. Complex polarizations.

Let P be a complex polarization of Ω_λ, which is translation invariant. Then P is generated by a constant vector field $v = \partial_x + \tau\partial_y$ where τ is any non-real complex number.

The functions satisfying the equation $vF = 0$ can be described quite explicitly. Namely, we introduce on Ω_λ a complex coordinate w so that the field v takes the form $v = \partial_{\bar{w}}$. Solving the system

$$w'_x + \tau w'_y = 0, \qquad w'_x + \bar{\tau} w'_y = 1$$

we obtain $w := \frac{\tau x - y}{\tau - \bar{\tau}}$. In particular, for $\tau = i$ we have $w = \frac{1}{2}(x + iy)$.

Therefore, the solutions to the equation $vF = 0$ are exactly the holomorphic functions of w. They form a maximal abelian subalgebra in $C^\infty(\Omega_\lambda)$ with respect to the Poisson bracket.

6.3. Discrete polarizations.

We have seen that the real polarizations of $\mathfrak{h}$ are related to maximal abelian connected subgroups in H. The complex polarizations are related to analogous subgroups in the complexification $H_\mathbb{C} = \exp \mathfrak{h}_\mathbb{C}$.

There is one more class of groups which are abelian modulo the kernel of a given unirrep π_λ of H. Namely, in exponential coordinates $g(a, b, c) = \exp(aX + bY + cZ)$, such a group is given by the condition:

$$a \in \alpha \cdot \mathbb{Z}, \qquad b \in \beta \cdot \mathbb{Z}, \qquad c \in \mathbb{R} \qquad \text{where } \alpha \cdot \beta \cdot \lambda = 1.$$

This polarization was used in Section 2.5 for constructing the third realization of CCR. It has an important analog in the representation theory of p-adic groups (see [**LV**]).

The Orbit Method for Nilpotent Lie Groups

The class of nilpotent Lie groups is the ideal situation where the orbit method works perfectly. It allows us to get simple and visual answers to all the important questions of representation theory. You can find the corresponding "User's Guide" in the Introduction to this book.

The nilpotent case can also serve as a model for more general and sophisticated theories considered in subsequent chapters.

All results of this chapter were obtained in [**Ki1**] with one exception: the fact that the bijection $\widehat{G} \longleftrightarrow \mathcal{O}(G)$ is a homeomorphism was proved later in [**Br**].

1. Generalities on nilpotent Lie groups

Here we list the basic properties of nilpotent Lie groups and Lie algebras that we use in this lecture. The proofs can be found in [**Bou**], [**J**], or derived from Appendix III.1 and Appendix III.2.

Definition 1. A Lie algebra $\mathfrak{g}$ is called a **nilpotent Lie algebra** if it possesses the properties listed in the proposition below.

Proposition 1. *The following properties of a Lie algebra $\mathfrak{g}$ are equivalent:*

a) *There exists a sequence of ideals in $\mathfrak{g}$*

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g} \tag{1}$$

such that

$$[\mathfrak{g}, \mathfrak{g}_k] \subset \mathfrak{g}_{k-1}, \quad 1 \leq k \leq n. \tag{2}$$

b) *The same as* a) *with the additional property* $\dim \mathfrak{g}_k = k$.

c) *For any* $X \in \mathfrak{g}$ *the operator* $\operatorname{ad} X$ *is nilpotent:* $(\operatorname{ad} X)^N = 0$ *for* N *big enough.*

d) $\mathfrak{g}$ *has a matrix realization by strictly upper triangular matrices* $X = \|X_{ij}\|$, *i.e. such that* $X_{ij} = 0$ *for* $i \geq j$. □

Definition 2. A connected Lie group G is called a **nilpotent Lie group** if its Lie algebra $\mathfrak{g}$ is nilpotent.

Proposition 2. *The following properties of a connected Lie group are equivalent:*

a) G *is a nilpotent Lie group.*

b) *There exists a sequence of connected normal subgroups*

$$\{e\} = G_0 \subset G_1 \subset \cdots \subset G_{n-1} \subset G_n = G \tag{3}$$

such that G_{k+1}/G_k *is in the center of* G/G_k.

c) *Same as* b) *with the additional property* $\dim G_k = k$, $1 \leq k \leq n$.

d) G *has a matrix realization by upper triangular matrices of the form* $g = \|g_{ij}\|$ *satisfying*

$$g_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i > j. \end{cases}$$

□

The minimal n for which the sequence (1) (respectively (3)) exists is called the **nilpotency class** of $\mathfrak{g}$ (resp. G). The abelian Lie groups have nilpotency class 1 while the generalized Heisenberg groups H_n have nilpotency class 2.

We also use the following features of nilpotent Lie groups.

Proposition 3. *Let* G *be a connected and simply connected nilpotent Lie group. Then*

a) *the exponential map* $\exp\colon \mathfrak{g} \to G$ *is a diffeomorphism that establishes a bijection between subalgebras* $\mathfrak{h} \subset \mathfrak{g}$ *and closed connected subgroups* $H \subset G$;

b) *in exponential coordinates the group law is given by polynomial functions of degrees not exceeding the nilpotency class;*

c) G *is unimodular and a two-sided invariant measure* dg *is just the Lebesgue measure* $d^n x = |dx_1 \wedge \cdots \wedge dx_n|$ *in exponential coordinates;*

d) *for any* $F \in \mathfrak{g}^*$ *we have* $Stab(F) = \exp stab(F)$. *Hence,* $Stab(F)$ *is connected and the coadjoint orbit* $\Omega_F \simeq G/Stab(F)$ *is simply connected.* □

2. Comments on the User's Guide

Here we explain in more detail the practical instructions given in the User's Guide and supply some additional information.

2.1. The unitary dual.

By definition, the **unitary dual** $\widehat{G}$ to a topological group G is the collection of all equivalence classes of **unirreps** of G. It is a topological space with the topology defined in Section 4.5. For a compact group G this topology is discrete. For discrete G the space $\widehat{G}$ is quasi-compact (i.e. compact but not necessarily Hausdorff).

For an abelian group G the set $\widehat{G}$ consists of all continuous homomorphisms $\chi : G \to \mathbb{T}^1$, called **multiplicative characters.**[1] If, moreover, G is locally compact, the set $\widehat{G}$ has itself a structure of a locally compact group. It is called the Pontryagin dual of G (cf. Chapter 2, Section 5.1).

For non-abelian groups the set $\widehat{G}$ is no longer a group, but some duality between G and $\widehat{G}$ still exists. For example, there is a generalized Fourier transform from $L^2(G)$ to $L^2(\widehat{G})$ (see Appendix V.1.5). The attempt to develop this duality and put G and $\widehat{G}$ on equal rights has led ultimately to the discovery of quantum groups (see [**Dr**] for historical comments).

2.2. The construction of unirreps.

A big part of the unirreps for all kinds of groups can be constructed using the induction procedure. For nilpotent Lie groups all unirreps can be constructed by applying the induction functor to 1-dimensional unirreps, i.e. multiplicative characters, of some connected subgroups $H \subset G$.

Lemma 1. *Any multiplicative character of a connected Lie subgroup $H \subset G$ has the form*

$$\rho_{F,H}(\exp X) = e^{2\pi i \langle F, X\rangle} \tag{4}$$

where F is a linear functional on $\mathfrak{g} = \mathrm{Lie}\,(G)$ with the property $F|_{[\mathfrak{h},\,\mathfrak{h}]} = 0$.

Proof. First note that any character ρ of a Lie group H is a smooth function on H (see Remark 4 in Appendix V.1.2). By Theorem 2 in Appendix III.1.3 we conclude that ρ is given by the equation (4) with $F \in \mathfrak{h}^*$. Since $p : \mathfrak{g}^* \to \mathfrak{h}^*$ is surjective, we can replace it with an $F \in \mathfrak{g}^*$.

The character ρ has the value 1 at all points of the commutator subgroup $[H,\, H]$. Therefore, the functional F must vanish on the commutator $[\mathfrak{h},\, \mathfrak{h}]$ of $\mathfrak{h}$. □

[1] The term "character" in representation theory is overloaded. Do not confuse the notion of a multiplicative character with notions of ordinary, generalized, distributional, and infinitesimal characters of Lie groups defined in Appendix V.1.3-4.

According to Rule 2 of the User's Guide, all unirreps of a nilpotent group G are among the representations $\operatorname{Ind}_H^G \rho_{F,H}$. We shall prove in Section 3 below that the representation $\pi_{F,H} = \operatorname{Ind}_H^G \rho_{F,H}$ is irreducible iff $\mathfrak{h}$ is a subalgebra of maximal dimension among the subalgebras subordinated to F. Actually, we have $\dim \mathfrak{h} = \frac{\dim G + \dim H}{2}$.

Moreover, the equivalence class of $\pi_{F,H}$ depends only on the coadjoint orbit $\Omega \subset \mathfrak{g}^*$ that contains F.

2.3. Restriction-induction functors.

The statements of Rules 3 and 4 of the User's Guide can be formulated a bit more precisely. Let π_Ω denote the unirrep of G corresponding to the orbit $\Omega \subset \mathfrak{g}^*$, and let ρ_ω denote the unirrep of H corresponding to the orbit $\omega \subset \mathfrak{h}^*$.

Denote by S_Ω the set of H-orbits in $p(\Omega) \subset \mathfrak{h}^*$ and by S^ω the set of G-orbits which have non-empty intersection with $p^{-1}(\omega) \subset \mathfrak{g}^*$. For nilpotent Lie groups the sets S_Ω and S^ω are finite unions of smooth manifolds, so they carry a natural equivalence class of measures defined by differential forms of top degree. We denote these measures by $d\omega$ and $d\Omega$ respectively.

The decomposition formulae look like

$$\operatorname{Res}_H^G \pi_\Omega = \int_{\omega \subset p(\Omega)} m(\Omega, \omega) \cdot \rho_\omega \, d\omega \tag{5}$$

and

$$\operatorname{Ind}_H^G \rho_\omega = \int_{p(\Omega) \supset \omega} m(\Omega, \omega) \cdot \pi_\Omega \, d\Omega. \tag{6}$$

By the Frobenius Duality Principle, the multiplicity function $m(\Omega, \omega)$ is the same in (5) and (6). For nilpotent Lie groups it takes only values $0, 1$, and ∞. It is convenient to write it in the form $m(\Omega, \omega) = \infty^{k(\Omega, \omega)}$ where ∞^k is interpreted as $0, 1$, or ∞ when k is negative, zero, or positive respectively.

According to the ideology of the orbit method, the integer k must be defined by geometry of the triple (Ω, ω, p). Recall that Ω and ω are symplectic manifolds and $p : \mathfrak{g} \to \mathfrak{h}$ is a Poisson map, so that $\Gamma_p \cap (\Omega \times \omega)$ is a coisotropic submanifold in the symplectic manifold $\left(\Omega \times \omega, \, \sigma_\Omega - \sigma_\omega\right)$. The precise answer is given below in Section 3.3.

2.4. Generalized characters.

For a nilpotent Lie group G the generalized characters are defined for any unirrep π as linear functionals on the space $\mathcal{M}(G)$ of all smooth rapidly decaying measures on G. In exponential coordinates the elements of $\mathcal{M}(G)$ look like $\mu = \rho(x) d^n x$ where $\rho \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space. If we fix a

smooth measure on G, e.g., the Haar measure dg, then generalized characters become the tempered distributions on G.

They are in general non-regular distributions. But the general theory of distribution guarantees that they can be written as generalized derivatives (of a certain order) of regular distributions. The order in question depends of the geometry of the orbit. To find the explicit form of this dependence is an open problem.

2.5. Infinitesimal characters.

In the paper [**Di1**] Dixmier proved that for a nilpotent Lie group G the algebra $\left(Pol(\mathfrak{g}^*)\right)^G$ of G-invariant polynomials on $\mathfrak{g}^*$ is always a polynomial algebra in a finite number of independent homogeneous generators $P_1, \ldots, P_k$. We can also assume that they are real-valued polynomials. So, the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ is a polynomial algebra with homogeneous Hermitian generators $A_j = (2\pi i)^{\deg P_j}\mathbf{sym}(P_j)$, $1 \leq j \leq k$.

For any unirrep π of G its infinitesimal character I_π is defined by the real numbers $c_j = I_\pi(A_j)$. If π_Ω is the unirrep corresponding to a coadjoint orbit Ω, then Rule 7 of the User's Guide claims that $c_j = P_j(\Omega)$. (The right-hand side makes sense because P_j is constant along Ω.)

Note that both the infinitesimal character $I_{\pi_\Omega} : Z(\mathfrak{g}) \to \mathbb{C}$ and the evaluation map $\mathrm{ev}_\Omega : \left(Pol(\mathfrak{g}^*)\right)^G \to \mathbb{C}$ are algebra homomorphisms. It follows that for any nilpotent Lie algebra $\mathfrak{g}$ the map $\mathbf{sym} : \left(Pol(\mathfrak{g}^*)\right)^G \to Z(\mathfrak{g})$ is an algebra homomorphism. This property is not true for general Lie groups.

2.6. Functional dimension.

We say that a unirrep (π, V) of a Lie group G has the **functional dimension** n if the Hilbert space V is naturally realized by functions of n variables. Here the word "naturally" means that the space V^∞ consists of smooth functions on which elements of $U(\mathfrak{g})$ act as differential operators.

For nilpotent groups we can say more. The space V^∞ can always be identified with the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ so that the image of $U(\mathfrak{g})$ is the algebra W_n of all differential operators with polynomial coefficients (see Section 4.3).

The correctness of the definition of the functional dimension is based on the following fact.

Theorem 1 (see [**GK**]). *The algebra W_n has GK-dimension $2n$, hence W_n and $W_{n'}$ are not isomorphic for $n \neq n'$.* □

We recall that the **GK-dimension** of a non-commutative algebra $\mathcal{A}$ is defined as follows. Let $\alpha = \{a_1, \ldots, a_k\}$ be any finite subset in $\mathcal{A}$. Let $[\alpha]^N$ denote the subspace in $\mathcal{A}$ spanned by all monomials in $a_1, \ldots, a_k$ of degree $\leq N$.

Then we put

$$\dim_{GK} \mathcal{A} = \sup_{\alpha} \lim_{N\to\infty} \frac{\log \dim [\alpha]^N}{\log N}. \tag{7}$$

This definition generalizes the notions of Krull dimension for commutative rings and of transcendence degree for quotient fields.

2.7. Plancherel measure.

Let G be a unimodular topological group with bi-invariant measure dg. In the space $L^2(G, dg)$ the natural unitary representation of the group $G\times G$ is defined. Namely,

$$\bigl(\pi(g_1, g_2)f\bigr)(g) = f(g_1^{-1}gg_2). \tag{8}$$

This representation is nothing but $\operatorname{Ind}_G^{G\times G} 1$ where G is considered as the diagonal subgroup in $G \times G$. Therefore, according to Rules 3 and 8 of the User's Guide, it decomposes into unirreps of the form $\pi_{\Omega,-\Omega} \simeq \pi_\Omega \times \pi_\Omega^*$.

Let V_Ω be the representation space for π_Ω. The representation space $V_{\Omega,-\Omega}$ for $\pi_\Omega \times \pi_\Omega^*$ is $V_\Omega \boxtimes V_\Omega^*$, i.e. the space of all Hilbert-Schmidt operators in V_Ω. The representation π acts as

$$\pi(g_1, g_2)A = \pi_\Omega(g_1) \cdot A \cdot \pi_\Omega(g_2)^{-1}.$$

The explicit isomorphism between $L^2(G, dg)$ and the continuous sum of Hilbert spaces $V_\Omega \boxtimes V_\Omega^*$, $\Omega \in \mathcal{O}(G)$, plays the role of the **non-commutative Fourier transform**. It looks as follows:

(9)

$$f \mapsto \pi_\Omega(f) = \int_G f(g)\pi_\Omega(g)dg, \qquad f(g) = \int_{\mathcal{O}(G)} \operatorname{tr}\bigl(\pi_\Omega(f)\pi_\Omega(g)^{-1}\bigr)d\mu(\Omega),$$
$$\|f\|^2_{L^2(G, dg)} = \int_{\mathcal{O}(G)} \|\pi_\Omega(f)\|_2^2 d\mu(\Omega).$$

The measure μ entering these formulae is called the **Plancherel measure** on the unitary dual $\widehat{G}$, which here is identified with the set $\mathcal{O}(G)$ of coadjoint orbits.

Note that this Fourier transform keeps some features of the classical Fourier transform on abelian groups. For example, left and right shifts go

under this transform to left and right multiplication by an operator-valued function:

$$\pi_\Omega(L_{g_1}R_{g_2}f) = \pi_\Omega(g_1)^{-1}\pi_\Omega(f)\pi_\Omega(g_2).$$

Another interpretation of the Plancherel measure can be obtained if we put $g = e$ in the second equation in (9). We get

$$\delta(g) = \int_{\mathcal{O}(G)} \chi_\Omega(g)\, d\mu(\Omega). \tag{10}$$

So, the Plancherel measure gives the explicit decomposition of the delta-function concentrated at the unit into characters of unirreps.

After the Fourier transform in canonical coordinates this relation acquires a geometrically transparent form. Namely, it gives a decomposition of the Lebesgue measure on $\mathfrak{g}^*$ into canonical measures on coadjoint orbits.

3. Worked-out examples

In this section we show how to apply the orbit method to a concrete nilpotent Lie group. The simplest example is that of the Heisenberg group. It was already treated in the previous chapter and will be mentioned again.

Here we choose the next, more complicated and more typical case. Among 4-dimensional nilpotent Lie algebras there exists a unique Lie algebra $\mathfrak{g}$ that cannot be split into a direct sum of ideals.[2] It has a basis $\{X_1, X_2, X_3, X_4\}$ with the commutation relations (we list only non-zero commutators):

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4. \tag{11}$$

This Lie algebra admits the following upper-triangular matrix realization:

$$X = \sum_{i=1}^{4} x^i X_i = \begin{pmatrix} 0 & x^1 & 0 & x^4 \\ 0 & 0 & x^1 & x^3 \\ 0 & 0 & 0 & x^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding matrix Lie group is denoted by G. A general element $g \in G$ in exponential parametrization looks as follows:

$$g(x^1, x^2, x^3, x^4) = \exp X = \begin{pmatrix} 1 & x^1 & \frac{(x^1)^2}{2} & x^4 + \frac{x^1x^3}{2} + \frac{(x^1)^2x^2}{6} \\ 0 & 1 & x^1 & x^3 + \frac{x^1x^2}{2} \\ 0 & 0 & 1 & x^2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

[2] Representations of this Lie algebra are used in the theory of an anharmonic oscillator (see [**Kl**]) and were first described in [**Di1**].

3.1. The unitary dual.

The first question of representation theory for a given group G is the description of its **unitary dual** $\widehat{G}$, i.e. of the collection of equivalence classes of unirreps.

There is a natural non-Hausdorff topology in $\widehat{G}$ which can be defined in three different but equivalent ways (see, e.g., [**Di3**] or [**Ki2**]). We explain the most convenient definition later in Section 4.5.

We shall see below that for a nilpotent Lie group G the set $\widehat{G}$ is always a finite union of smooth manifolds that are glued together in an appropriate way.

The exponential coordinate system, being very useful in theory, is often not the best for computations. In our case we get simpler expressions if we use the so-called canonical coordinates of the second kind and put

$$\begin{aligned} g'(a^1, a^2, a^3, a^4) &:= \exp a_4 X_4 \cdot \exp a_3 X_3 \cdot \exp a_2 X_2 \cdot \exp a_1 X_1 \\ &= \begin{pmatrix} 1 & a^1 & \frac{(a^1)^2}{2} & a^4 \\ 0 & 1 & a^1 & a^3 \\ 0 & 0 & 1 & a^2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The element $F \in \mathfrak{g}^*$ with coordinates $\{X_1, X_2, X_3, X_4\}$ can be written as a lower triangular matrix of the form

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ X_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X_4 & X_3 & X_2 & 0 \end{pmatrix} \qquad \text{with} \quad \langle F, X \rangle = \operatorname{tr}(FX) = \sum_{i=1}^{4} x^i X_i.$$

These matrices form a subspace $V \subset \mathrm{Mat}_4(\mathbb{R})$ and the projection $p : \mathrm{Mat}_4(\mathbb{R}) \to \mathfrak{g}^*$ parallel to $\mathfrak{g}^\perp$ has the form

$$p(\| A_{ij} \|) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ A_{21} + A_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{41} & A_{42} & A_{43} & 0 \end{pmatrix}.$$

Note that the subspace V is not the only possible choice of a subspace of lower triangular matrices that is transversal to $\mathfrak{g}^\perp$. We could, for instance, take as V the set of matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & X_1 & 0 & 0 \\ X_4 & X_3 & X_2 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2}X_1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}X_1 & 0 & 0 \\ X_4 & X_3 & X_2 & 0 \end{pmatrix}.$$

We leave it to the reader to compare the resulting formulae for the coadjoint action with those shown below.

Exercise 1. Explicitly compute the coadjoint action of G on $\mathfrak{g}^*$ in the above coordinates using formula (4) from Chapter 1.

Answer:

(12)

$$K\big(g(x^1,x^2,x^3,x^4)\big)(X_1,X_2,X_3,X_4)$$
$$=\Big(X_1+x^2X_3+(x^3-\frac{x^1x^2}{2})X_4, X_2-x^1X_3+\frac{(x^1)^2}{2}X_4, X_3-x^1X_4, X_4\Big);$$
$$K\big(g'(a^1,a^2,a^3,a^4)\big)(X_1,X_2,X_3,X_4)$$
$$=\Big(X_1+a^2X_3+(a^3-a^1a^2)X_4, X_2-a^1X_3+\frac{(a^1)^2}{2}X_4, X_3-a^1X_4, X_4\Big).$$

For future use we describe here all polynomial invariants of the coadjoint action. One of them is obvious: $P(F)=X_4$. This is immediately seen from (12) but can also be explained more conceptually. Namely, the basic element $X_4\in\mathfrak{g}$ belongs to the center of $\mathfrak{g}$. Hence, the corresponding coordinate X_4 on $\mathfrak{g}^*$ is unchanged under the coadjoint action of G.

Note that P cannot be the only invariant, because the generic orbit cannot have codimension 1. (Recall that coadjoint orbits are even-dimensional and $\dim\mathfrak{g}=4$.) To find another invariant, we follow the scheme described in Chapter 1 (see I.3.1).

Let us consider the plane S given by two linear equations $X_1=X_3=0$. It is clear from equation (5) that almost all orbits meet this plane.[3] An easy computation shows that the orbit Ω passing through the point $(X_1,\ X_2,\ X_3,\ X_4)$ with $X_4\neq 0$ intersects the plane S in the single point with coordinates $(0,\ X_2-\frac{X_3^2}{2X_4},\ 0,\ X_4)$. It follows that the quantity $R(F)=X_2-\frac{X_3^2}{2X_4}$ is a rational invariant. So, the second polynomial invariant is $Q(F)=2R(F)P(F)=2X_2X_4-X_3^2$.

Lemma 2. *Let $Pol(\mathfrak{g}^*)^G$ be the algebra of all G-invariant polynomial functions on $\mathfrak{g}^*$. Then*

$$Pol(\mathfrak{g}^*)^G\simeq S(\mathfrak{g})^G=\mathbb{C}[P,\,Q]\quad \textit{with } P=X_4,\ Q=2X_2X_4-X_3^2.$$

Proof. The first isomorphism holds for any Lie algebra (see Appendix III). To prove the second equality we consider the common level set $S_{p,q}$ of the two invariants P, Q given by

$$S_{p,q}=\{F\in\mathfrak{g}^*\mid P(F)=p,\ Q(F)=q\}. \tag{13}$$

[3] E.g., all orbits with $X_4\neq 0$ have this property.

Exercise 2. Check that the set $S_{p,q}$

• for $p \neq 0$, is a single orbit $\Omega_{p,q}$ passing through the point $(0,\, \frac{q}{2p},\, 0,\, p)$;

• for $p = 0$, $q < 0$, splits into two orbits $\Omega^{\pm}_{0,q}$ passing through the points $(0,\, 0,\, \pm\sqrt{-q},\, 0)$;

• for $p = 0$, $q > 0$, is empty;

• for $p = q = 0$, splits into 0-dimensional orbits $\Omega^{c_1, c_2}_{0,0} = \{(c_1,\, c_2,\, 0,\, 0)\}$, which are fixed points of the coadjoint action. ♣

Now, let $\Phi \in Pol(\mathfrak{g}^*)^G$ be any invariant polynomial on $\mathfrak{g}^*$. Then it takes a constant value on each coadjoint orbit and, in particular, on every set $S_{p,q}$ with $p \neq 0$.

On the other hand, the restriction of Φ on the plane S is a polynomial ϕ in coordinates X_2 and X_4. Compare Φ with the function $\phi(\frac{Q}{2P},\, P)$. They coincide on $S \cap \{X_4 \neq 0\}$ and are both G-invariant. Therefore, they coincide on $\mathfrak{g}^* \cap \{P \neq 0\}$, hence everywhere. Thus, Φ has the form $\Phi = \frac{A(P,Q)}{P^N}$ for some polynomial A and some integer N. But Φ, being polynomial on $\mathfrak{g}$, is regular on the hyperplane $P = 0$. It follows that $\Phi \in \mathbb{C}[P,\, Q]$. □

The final description of the topological space $\mathcal{O}(G)$ looks as follows. Take a real plane $\mathbb{R}^2_{p,q}$, delete the line $p = 0$, and glue to the remaining set

a) two points instead of each point of the deleted ray $p = 0,\ q < 0$,

b) a whole 2-plane instead of the deleted origin.

The topology of $\mathcal{O}(G)$ is the standard quotient topology: a set of orbits is open iff the union of all orbits from this set is open in $\mathfrak{g}^*$. In particular, the limit[4] of the sequence $\{\Omega_{\epsilon_n, c}\}$ where $\epsilon_n \to 0$ is

a) two points $\Omega^{\pm}_{0,c}$ for $c < 0$;

b) the whole plane of zero-dimensional orbits $\Omega^{c_3, c_4}_{0,0}$ for $c = 0$;

c) no limit for $c > 0$.

According to Rule 1 of the User's Guide, there is a homeomorphism, i.e. a bicontinuous bijection, between the set $\widehat{G}$ of all unirreps (considered up to equivalence) and the set $\mathcal{O}(G)$ of coadjoint orbits.

In this example the $K(G)$-invariant polynomials separate the generic orbits $\Omega_{p,q}$ but do not separate the special orbits $\Omega^{\pm}_{0,q}$ and the degenerate orbits $\Omega^{c_1, c_2}_{0,0}$.

3.2. Construction of unirreps.

According to Rule 2 of the User's Guide, the representation π_Ω is induced from a 1-dimensional unirrep $\rho_{F,H}$ of an appropriate subgroup $H \subset G$.

[4]Recall that since the space in question is non-Hausdorff, the limit is non-unique.

To get the explicit formula we have to make the following steps:

(i) pick any point $F \in \Omega$;

(ii) find a subalgebra $\mathfrak{h}$ of maximal dimension which is subordinate to F, i.e. such that $B_F \big|_{\mathfrak{h}} = 0$, or $F \big|_{[\mathfrak{h},\mathfrak{h}]} = 0$;

(iii) take a subgroup $H = \exp \mathfrak{h}$ and define the 1-dimensional unirrep $\rho_{F,H}$ of H by the formula

$$\rho_{F,H}(\exp X) = e^{2\pi i \langle F,\, X\rangle}; \tag{14}$$

(iv) choose a section $s: \ X = H\backslash G \to G$ and solve the master equation

$$s(x)\, g = h(x,\, g)\, s(x \cdot g); \tag{15}$$

(v) compute the measure μ_s on X and write the final formula

$$\big(\pi_\Omega(g) f\big)(x) = \rho_{F,H}\big(h(x,\, g)\big) f(x \cdot g) \tag{16}$$

for the realization of π_Ω in the space $L^2(X,\, \mu_s)$.

Remark 1. In our case (i.e. when G is a connected and simply connected nilpotent Lie group G and H is a closed connected subgroup) the homogeneous space $X = H\backslash G$ can be identified with $\mathbb{R}^k$, $k = \dim G - \dim H$, as follows.

Choose any k-dimensional subspace $\mathfrak{p} \subset \mathfrak{g}$ that is transversal to $\mathfrak{h} = \mathrm{Lie}(H)$. Then any point of G can be uniquely written as $g = h \cdot \exp p$, $p \in \mathfrak{p}$. So, any coset $x = Hg$ has a unique representative of the form $\exp p$, $p \in \mathfrak{p}$, hence, can be identified with $p \in \mathfrak{p}$.

If we define the section $s : X \to G$ by $s(p) = \exp p$, one can check that the measure μ_s is just the Lebesgue measure on $\mathfrak{p}$.

♡

Here we make the explicit computations for all unirreps of the 4-dimensional Lie group G discussed in the previous section. We keep the notation from there.

Let us start with zero-dimensional orbits. Here the form B_F is identically zero, since $\operatorname{rk} B_F = \dim \Omega = 0$. Hence, $\mathfrak{h} = \mathfrak{g}$, $H = G$, and $\pi_\Omega = \rho_{F,H}$.

Thus, the 1-dimensional representation π_Ω of G associated with the single-point orbit $\Omega = \{F\}$ is given by

$$\pi_\Omega(\exp X) = e^{2\pi i \, \langle F,\, X\rangle}. \tag{17}$$

In our case, we associate the representation

$$\pi_{0,0}^{c_1,c_2}\big(g(a_1,\, a_2,\, a_3,\, a_4)\big) = e^{2\pi i\,(c_1a_1+c_2a_2)} \tag{18}$$

to the orbit $\Omega_{0,0}^{c_1,c_2} = \{(c_1,\, c_2,\, 0,\, 0)\}$.

All remaining orbits are 2-dimensional. They correspond to infinite-dimensional representations which have **functional dimension** one (see Rule 9 of the User's Guide).

The latter statement, roughly speaking, means that the representation in question is naturally realized in a space of functions of one variable. Indeed, each of these representations, according to Rule 2 of the User's Guide, can be obtained by the induction procedure from a 1-dimensional representation of a subgroup H of codimension 1. Therefore, it acts in the space of sections of a line bundle L over a 1-dimensional manifold. Such a section locally is given by a function of one variable.

The precise definition of the notion of functional dimension was discussed in Section 2.6 above (see also the Comments on the User's Guide).

So, we have to find a 3-dimensional subalgebra $\mathfrak{h} \subset \mathfrak{g}$ that is subordinate to F. The general procedure is described in Lemma 9 of Chapter 1, Section 5.2. In fact, for our group G we can choose the same subalgebra $\mathfrak{h}$ for all remaining unirreps.

Namely, take as $\mathfrak{h}$ the linear span of X_2, X_3, X_4. The point is that $\mathfrak{h}$, being abelian, is subordinate to any functional $F \in \mathfrak{g}^*$ and has the right dimension.

Remark 2. Note that the observed phenomenon is not a general rule. For other nilpotent groups it may happen that we need to use infinitely many different subalgebras in the role of $\mathfrak{h}$ to construct all the unirreps.

Consider for example the universal nilpotent Lie algebra $\mathfrak{g}$ of nilpotency class 2 with 3 generators. This 6-dimensional Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ has a basis $\{X_i,\, Y^j\}_{1\le i,j\le 3}$ with commutation relations

$$[X_i,\, X_j] = \epsilon_{ijk}Y^k, \qquad [X_i,\, Y^j] = [Y^j,\, Y^k] = 0.$$

The generic orbit in $\mathfrak{g}^*$ is a 2-dimensional plane defined by the equations

$$Y^j = c^j,\ 1 \le j \le 3, \qquad \text{and} \qquad c^iX_i = c.$$

The kernel $stab(F)$ of the form B_F is spanned by all Y^j and $X = c^iX_i$. The role of $\mathfrak{h}$ can be played by any subalgebra of codimension 1 that contains $stab(F)$. It remains to observe that a finite number of 2-planes cannot cover the whole $\mathbb{R}^3$. ♡

To construct the induced representation $\operatorname{Ind}_H^G \rho_{F,H}$, we apply the standard procedure described briefly in the beginning of this section and in full detail in Appendix V.2.

In our special case it is convenient to identify the homogeneous space $M = H\backslash G$ with $\mathbb{R}$ and define the section $s : \mathbb{R} \to G$ by $s(x) = \exp\, xX_1$. Then the measure μ_s will be the standard Lebesgue measure $|dx|$ on $\mathbb{R}$.

We have to solve the master equation (15), which in our case takes the form

$$\begin{pmatrix} 1 & x & \frac{x^2}{2} & 0 \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a^1 & \frac{(a^1)^2}{2} & a^4 \\ 0 & 1 & a^1 & a^3 \\ 0 & 0 & 1 & a^2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & h^4 \\ 0 & 1 & 0 & h^3 \\ 0 & 0 & 1 & h^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & y & \frac{y^2}{2} & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with given a^i and x and unknown h^i and y. The solution is

$$y = x + a^1, \quad h^2 = a^2, \quad h^3 = a^3 + a^2 x, \quad h^4 = a^4 + a^3 x + \frac{1}{2} a^2 x^2.$$

The next step is to write the explicit formula for $\rho_{F,H}$.

For generic orbits $\Omega_{p,q}$ with $p \neq 0$ we take the representative $F = (0, \frac{q}{2p}, 0, p)$ while for the remaining orbits $\Omega^{\pm}_{0,q}$, $q < 0$, we take the representative $F = (0, 0, \pm\sqrt{-q}, 0)$.

We obtain:

$$\rho_{F,H}(h) = e^{2\pi i \langle F, \log h\rangle} = \begin{cases} e^{2\pi i (a^2 \frac{q}{2p} + a^4 p)} & \text{for } \Omega_{p,q}, \\ e^{\pm 2\pi i a^3 \sqrt{-q}} & \text{for } \Omega^{\pm}_{0,q}. \end{cases}$$

So, we come to the following final results:

$$(19) \quad \big(\pi_{p,q}\big(g'(a^1, a^2, a^3, a^4)\big) f\big)(x) = e^{2\pi i \left(p(a^4 + a^3 x + \frac{1}{2} a^2 x^2) + \frac{q}{2p} a^2\right)} f(x + a^1)$$

and

$$(20) \quad \big(\pi^{\pm}_{0,q}\big(g'(a^1, a^2, a^3, a^4)\big) f\big)(x) = e^{\pm 2\pi i \sqrt{-q}(a^3 + a^2 x)} f(x + a^1).$$

Here the notation g' reminds us that we use the canonical coordinates of the second kind.

From (19) and (20) we easily derive formulas for the representations of $U(\mathfrak{h})$. The results are collected in Table 1. From these results we conclude that the space H^∞ of smooth vectors for the representations (19) and (20) coincides with the Schwartz space $\mathcal{S}(\mathbb{R}^1)$, where $U(\mathfrak{h})$ acts as the algebra of differential operators with polynomial coefficients. Later, in Section 4.3, we prove that this is a common feature of unirreps for all nilpotent Lie groups.

Table 1. The images of basic elements of $\mathfrak{g}$ in all unirreps of G.

	$\pi_{p,q}$	$\pi^{\pm}_{0,q}$	$\pi^{c_1,c_2}_{0,0}$
X_1	$\dfrac{d}{dx}$	$\dfrac{d}{dx}$	$2\pi i c_1$
X_2	$\pi i p x^2 + \dfrac{\pi i q}{p}$	$\pm\, 2\pi i x\sqrt{-q}$	$2\pi i c_2$
X_3	$2\pi i p x$	$\pm\, 2\pi i\sqrt{-q}$	0
X_4	$2\pi i p$	0	0

Note that the commutation relations (1) are obviously satisfied.

3.3. Restriction functor.

Rule 3 of the User's Guide for simply connected nilpotent Lie groups can be formulated in a more precise form than was stated in the Introduction. We mentioned in the Comments on the User's Guide that not only the spectrum of $\mathrm{Res}\,^G_H \pi_\Omega$ but also the multiplicity function can be described in terms of orbits. It turns out that for nilpotent Lie groups it takes only three values: 0, 1, ∞.

To formulate the result, we introduce some notation. Let Ω be a coadjoint orbit in $\mathfrak{g}^*$, and let $\omega \subset p(\Omega)$ be a coadjoint orbit in $\mathfrak{h}^*$. Denote by Γ_p the graph of the projection $p:\ \mathfrak{g}^* \to \mathfrak{h}^*$. It is a subset in $\mathfrak{g}^* \times \mathfrak{h}^*$ consisting of all pairs $\big(F,\, p(F)\big)$, $F \in \mathfrak{g}^*$. Define the number $k(\Omega,\, \omega)$ by the formula

$$k(\Omega,\, \omega) = \dim\, (\Gamma_p \cap\, (\Omega \times \omega)) - \frac{1}{2}(\dim\, \Omega + \dim\, \omega). \tag{21}$$

The more precise form of Rule 3 has the form

$$\mathrm{Res}\,^G_H\, \pi_\Omega = \int_{\mathcal{O}(H)} m(\Omega,\, \omega) \cdot \pi_\omega\, d\omega \tag{22}$$

where

$$m(\Omega,\, \omega) = \infty^{k(\Omega,\, \omega)}. \tag{23}$$

Here we interpret $m = \infty^k$ as 0, 1, or ∞ according to whether k is negative, zero, or positive. In particular, $m(\Omega, \omega) = 0$ if $\omega \not\subset p(\Omega)$.

We continue to explore the group G from Sections 2.1 and 2.2. In this section we consider the restriction of the generic representation $\pi_{p,q}$ of G to different subgroups.

Consider first the abelian subalgebra $\mathfrak{a}$ spanned by X_2, X_3, X_4. The representation $\pi_{p,q}$ after restriction to the subgroup $A = \exp \mathfrak{a}$ must split into 1-dimensional unirreps of A. These unirreps have the form

$$u_{\lambda,\mu,\nu}\big(\exp(a^2X_2 + a^3X_3 + a^4X_4)\big) = e^{2\pi i(a^2\lambda+a^3\mu+a^4\nu)}.$$

Formula (19) shows that Res_A^G is a continuous sum of a 1-parametric family of unirreps $u_{\lambda,\mu,\nu}$ with

$$\lambda = \frac{px^2}{2} + \frac{q}{2p}, \quad \mu = px, \quad \nu = p, \qquad x \in \mathbb{R}. \tag{24}$$

Thus,

$$\mathrm{Res}_H^G\,\pi_{p,q} = \int_{\mathbb{R}} \pi_{\omega_x}\,dx$$

where $\omega_x = \{\frac{px^2}{2} + \frac{q}{2p},\ px,\ p\}$. We see that $m(\Omega,\,\omega_x) = 1$ for all $x \in \mathbb{R}$.

On the other hand, the projection of the orbit $\Omega_{p,q} \subset \mathfrak{g}^*$ to $\mathfrak{a}^*$ is the parabola given in coordinates $\lambda = X_2$, $\mu = X_3$, $\nu = X_4$ by equations $\nu = p$, $2\lambda\nu - \mu^2 = q$.

But (24) is exactly the parametric equation for this parabola. Moreover, in our case $k(\Omega_{p,q},\,\omega_x) = \frac{1}{2}(2+0) - 1 = 0$ for all $x \in \mathbb{R}$. So, Rule 3 in the precise form (22), (23) gives the correct answer.

Consider now the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ generated by X_1, X_3, X_4, which is isomorphic to the Heisenberg Lie algebra. Let $H = \exp \mathfrak{h}$ be the corresponding subgroup. We denote by $h(a,\,b,\,c)$ the element of the group H given by $\exp aX_1 \exp bX_3 \exp cX_4$.

The representations and coadjoint orbits for H were considered in detail in the previous chapter. In particular, we showed that H has a series of infinite-dimensional unirreps which depend on one real parameter λ and act in $L^2(\mathbb{R},\,dx)$ by the formula[5]

$$\big(\pi_\lambda(h(a,\,b,\,c))f\big)(x) = e^{2\pi i\lambda(bx+c)}f(x+a).$$

The projection of $\mathfrak{g}^*$ to $\mathfrak{h}^*$ is the natural map from $\mathbb{R}^4$ with coordinates X_1, X_2, X_3, X_4 to $\mathbb{R}^3$ with coordinates X_1, X_3, X_4 (the projection parallel to the X_3-axis). The orbit $\Omega_{p,q}$ given by equations (13) is projected to the plane $X_4 = p$. The number $k(\Omega_{p,q},\,\omega_p)$ is equal to $\frac{1}{2}(2+2) - 2 = 0$.

On the other hand, from (19) we see that the restriction $\mathrm{Res}_H^G\,\pi_{p,q}$ is exactly π_λ with $\lambda = p$.

So, again Rule 3 gives the right answer.

[5]This formula follows also from Rule 2 of the User's Guide.

Exercise 3. Derive from the refined Rule 3 the fact that $\mathrm{Res}_H^G \pi_{0,q}^{\pm}$ is equivalent to π_λ with $\lambda = \pm\sqrt{-q}$. Then check it using formula (20). ♣

It would be interesting to investigate in more detail the properties of the number $k(\Omega, \omega)$. The following lemma is a step in this direction.

Lemma 3. *Assume that $p(\Omega)$ splits into an s-dimensional family of coadjoint orbits $\omega_x \subset \mathfrak{h}^*$, $x \in X$, of the same dimension $2m$ and that $p^{-1}\big(p(\Omega)\big) = \Omega + \mathfrak{h}^\perp$ splits into a t-dimensional family of coadjoint orbits $\Omega_y \subset \mathfrak{g}^*$, $y \in Y$, of the same dimension $2n$ and the same projection to $\mathfrak{h}^*$. Then for all $x \in X$, $y \in Y$ we have*

$$k(\Omega_y, \omega_x) = n - s - m = m + r - t - n \tag{25}$$

where $r = \dim \mathfrak{h}^\perp = \dim \mathfrak{g} - \dim \mathfrak{h}$. □

Note that these relations agree with the "naive" count of variables. For instance, if π_Ω acts in the space of functions of n variables, then the restricted representation acts in the same space. On the other hand, an s-dimensional family of unirreps with the functional dimension m needs only $s + m$ variables. So, we have $n - s - m$ extra variables and the multiplicity must be ∞^{n-s-m}.

3.4. Induction functor.

Here again we formulate a more precise result than Rule 4 of the User's Guide. Namely,

$$\mathrm{Ind}\,_H^G \pi_\omega = \int_{\Omega \subset p^{-1}(K(G)\omega)} m(\Omega, \omega)\pi_\Omega \, d\Omega \tag{26}$$

where the multiplicity function $m(\Omega, \omega)$ is the same as in (22).

A simplest variant of an induced representation of a Lie group G is a representation $\pi = \mathrm{Ind}_H^G 1$ in the natural L^2-space on a homogeneous manifold $M = H\backslash G$.

Our group G has the matrix realization with canonical coordinates a^1, a^2, a^3, a^4 described in Section 2.1. Therefore, it acts on the vector space $\mathbb{R}^4$ with coordinates $\{x^1, x^2, x^3, x^4\}$:

$$\begin{pmatrix} 1 & a^1 & \frac{(a^1)^2}{2} & a^4 \\ 0 & 1 & a^1 & a^3 \\ 0 & 0 & 1 & a^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \begin{pmatrix} x^1 + a^1x^2 + \frac{(a^1)^2}{2}x^3 + a^4x^4 \\ x^2 + a^1x^3 + a^3x^4 \\ x^3 + a^2x^4 \\ x^4 \end{pmatrix}.$$

This linear action preserves the last coordinate x^4 (because the matrices in question are strictly upper triangular with units on the main diagonal).

In geometric language this means that G preserves an affine hyperplane $x^4 = c$, hence acts by affine transformation of $\mathbb{R}^3$:

$$g'(a^1, a^2, a^3, a^4) \cdot \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x^1 + a^1x^2 + \frac{(a^1)^2}{2}x^3 + a^4c \\ x^2 + a^1x^3 + a^3c \\ x^3 + a^2c \end{pmatrix}.$$

Note that this is a left action.

Since the affine transformations in question are unimodular, we get a unitary representation π_c of G in $L^2(\mathbb{R}^3, d^3x)$:

$$\begin{aligned} &\big(\pi_c\big(g'(a^1, a^2, a^3, a^4)^{-1}\big)f\big)(x^1, x^2, x^3) \\ &\quad = f\left(x^1 + a^1x^2 + \frac{(a^1)^2}{2}x^3 + a^4c,\ x^2 + a^1x^3 + a^3c,\ x^3 + a^2c\right). \end{aligned}$$

(The inverse element $(g')^{-1}$ appears in the formula because we deal with a left action of G.)

The question is, what is the spectrum of this representation? Or, in other words, how does it decompose into irreducible components?

We consider our hyperplane $X_4 = c$ for $c \neq 0$ as a left homogeneous space $X = G/B$ where the subgroup B is the stabilizer of some point x_0 of the hyperplane.

Choosing $x_0 = 0$, the origin, we get $B = \exp\mathfrak{b} = \exp\{\mathbb{R}\cdot X_1\}$. Therefore, π_c is just the induced representation $\mathrm{Ind}_B^G 1$ where 1 denotes the trivial 1-dimensional unirrep of the subgroup B. In particular, π_c belongs to the same equivalence class for all $c \neq 0$.

According to Rule 4, to describe the spectrum of π we have to consider the projection $p\colon \mathfrak{g}^* \to \mathfrak{b}^*$, take the G-saturation of $p^{-1}(0)$, and decompose it to G-orbits. In our case p is just the coordinate projection onto the X_1-axis and $p^{-1}(0)$ is the hyperplane $X_1 = 0$. The G-saturation of this hyperplane contains all 2-dimensional orbits $\Omega_{p,q}$ and $\Omega_{0,q}^{\pm}$ and also a 1-parametric family of 0-dimensional orbits $\Omega_{0,0}^{0,c_2}$. Since in decomposition problems one can neglect the sets of measure 0, we can restrict ourselves to the representations $\pi_{p,q}$ corresponding to generic orbits. The final answer looks as follows:

$$\pi = \int \pi_{p,q} d\mu(p, q) \tag{27}$$

where μ is any measure equivalent to the Lebesgue measure on $\mathbb{R}^2_{p,q}$.

Remark 3. The space $L^2(\mathbb{R}^3, d^3x)$ has functional dimension 3 while the generic unirreps have functional dimension 1. So, it is natural that the decomposition involves a 2-parametric family of unirreps.

The decomposition (27) can be interpreted as a simultaneous diagonalization, or spectral decomposition, of the two Laplace operators $\Delta_i = \pi(A_i)$, $i = 1, 2$ (see the next section). ♡

Now we check this answer and give the explicit formula for the abstract decomposition (27). For any function $f \in L^2(\mathbb{R}^3, d^3x)$ we introduce the family of functions $\psi_{p,q} \in L^2(\mathbb{R}, dt)$:

$$\psi_{p,q}(t) = \widehat{f}\left(p,\, pt,\, \frac{p^2t^2+q}{2p}\right) \tag{28}$$

where $\widehat{f}$ is the Fourier transform of f.

Proposition 4. a) *The correspondence $f \longleftrightarrow \psi$ is invertible:*

$$f(x^1, x^2, x^3) = \int_{\mathbb{R}^2} \psi_{p,q}(t) e^{-2\pi i(px^1+ptx^2+\frac{p^2t^2+q}{2p}x^3)} \left|\frac{dp \wedge dq}{2}\right| \tag{29}$$

and unitary:

$$|f|^2_{L^2(\mathbb{R}^3,\, d^3x)} = \iint |\psi_{p,q}|^2_{L^2(\mathbb{R},\, dt)} \left|\frac{dp \wedge dq}{2}\right|. \tag{30}$$

b) *When f is transformed by the operator $\pi(g)$, $g \in G$, the corresponding $\psi_{p,q}$ is transformed by $\pi_{p,q}(g)$.* □

Note that in the concrete decomposition formula (30) the abstract measure $d\mu(p,q)$ from (27) takes the concrete form $\left| \frac{dp\wedge dq}{2} \right|$.

3.5. Decomposition of a tensor product of two unirreps.

Here, as an example, we compute the spectrum of the tensor product $\pi_{p,q} \otimes \pi^{\pm}_{0,r}$. According to Rule 5 of the User's Guide we have to consider the arithmetic sum of $\Omega_{p,q}$ and $\Omega^{\pm}_{0,r}$.

The generic points of these orbits have the coordinates $(X,\, \frac{q+Y^2}{2p},\, Y,\, p)$ and $(x,\, y,\, \pm\sqrt{-q},\, 0)$, respectively. It follows that the arithmetic sum is the hyperplane $X_4 = p$. This hyperplane is the union of all orbits $\Omega_{p,q}$ with given $p \neq 0$. So, the answer is: the spectrum consists of all representations $\pi_{p,q}$ with fixed p, and $\pi_{p,q} \otimes \pi^{\pm}_{0,r}$ is a multiple of $\int \pi_{p,q} d\mu(q)$.

The more delicate question about multiplicities can be answered by the count of functional dimensions: for the tensor product $\pi_{p,q} \otimes \pi^{\pm}_{0,r}$ the functional dimensions add to 1+1= 2. On the other hand, the continuous sum, or integral, of all $\pi_{p,q}$ with a fixed value of p also has the functional dimension 2. This suggests the equality

$$\pi_{p,q} \otimes \pi^{\pm}_{0,r} = \int \pi_{p,q} d\mu(q). \tag{31}$$

The abstract formula (31) has a concrete form (compare Proposition 4 above). It can be obtained directly if we look at the tensor product of the representations given by (19) and (20). We use the fact that the Hilbert tensor product of $L^2(\mathbb{R}, |dx|)$ and $L^2(\mathbb{R}, |dy|)$ is naturally isomorphic to $L^2(\mathbb{R}^2, |dx \wedge dy|)$.

So, our representation has the form

$$\begin{aligned}\text{(32)}\qquad &\Big(\big(\pi_{p,q} \otimes \pi^{\pm}_{0,r}\big)\big(g'(a^1, a^2, a^3, a^4)\big) f\Big)(x,\, y)\\ &\qquad = e^{2\pi i\{p(a^4+a^3x+\frac{1}{2}a^2x^2)+\frac{q}{2p}a^2 \pm \sqrt{-r}(a^3+a^2y)\}} f(x+a^1,\, y+a^1).\end{aligned}$$

Let us split the plane with coordinates x, y into the family of parallel lines $y = x + t$. Then the space $L^2(\mathbb{R}^2,\, dxdy)$ splits into the continuous direct sum, or direct integral, of smaller Hilbert spaces. Namely, for $f \in L^2(\mathbb{R}^2,\, |dx \wedge dy|)$ define the family of functions $\psi_t(x) = f(x,\, x+t)$.

It is clear that

$$\text{(33)}\qquad |f|^2_{L^2(\mathbb{R}^2,\, dxdy)} = \int_{\mathbb{R}} |\psi_t|^2_{L^2(\mathbb{R},\, dx)} dt$$

and that ψ_t transforms according to $\pi_{p,q'}$ for some q' when f transforms according to (32).

So, (33) shows that the representation (32) splits into a direct continuous sum of representations of type $\pi_{p,q'}$ with fixed p and various q'.

Exercise 4. Find the exact value q' corresponding to the line $y = x+t$.

Answer: $q' = q + 2tpr - 2r^2$. ♣

Exercise 5. Decompose the tensor product of two generic unirreps into unirreps.

Answer: $\pi_{p',\,q'} \otimes \pi_{p'',\,q''} = \begin{cases} \int_{-\infty}^{+\infty} \pi_{p'+p'',\,q}\, dq & \text{if } p'+p'' \neq 0, \\ \int_{-\infty}^{+\infty} (\pi^+_{0,\,r} \oplus \pi^-_{0,\,r})\, dr & \text{if } p'+p'' = 0. \end{cases}$ ♣

3.6. Generalized characters.

Here we shall compute the generalized character $\chi_{p,q}$ of the unirrep $\pi_{p,q}$.

According to Rule 6 of the User's Guide we have to compute the integral

$$\text{(34)}\qquad \chi_{p,q}(\exp X) = \int_{\Omega_{p,q}} e^{2\pi i\langle F, X\rangle} \sigma$$

as a generalized function on G in the exponential coordinate $X = \log g$.

We start with the calculation of the canonical form on $\Omega_{p,q}$ which we denote by $\sigma_{p,q}$. From (12) we get the following expression for the infinitesimal coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$:

$$K_*(X_1) = X_3\partial_2 + X_4\partial_3,\ K_*(X_2) = -X_3\partial_1,\ K_*(X_3) = -X_4\partial_1,\ K_*(X_4) = 0.$$

The orbit $\Omega_{p,q}$ is given by equations (13). We choose $x := X_1$ and $y := X_3$ as global coordinates on the orbit and obtain the parametric representation of $\Omega_{p,q}$:

$$X_1 = x, \quad X_2 = \frac{q+y^2}{2p}, \quad X_3 = y, \quad X_4 = p.$$

In terms of coordinates x, y the $\mathfrak{g}$-action on $\Omega_{p,q}$ takes the form

$$K_*(X_1) = p\partial_y, \quad K_*(X_2) = -y\partial_x, \quad K_*(X_3) = -p\partial_x, \quad K_*(X_4) = 0.$$

Definition 1 from Chapter 1 in this case implies

$$\sigma_{p,q} = \frac{dx \wedge dy}{p}. \tag{35}$$

We are now in a position to compute the integral (34). For

$$X = \begin{pmatrix} 0 & a_1 & 0 & a_4 \\ 0 & 0 & a_1 & a_3 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p & y & \frac{q+y^2}{2p} & 0 \end{pmatrix}$$

it can be written as

$$\iint e^{2\pi i(a_1 x + a_2 \frac{q+y^2}{2p} + a_3 y + a_4 p)} \frac{|dx \wedge dy|}{p}.$$

Using the well-known relations

$$\int_{\mathbb{R}} e^{2\pi i\, ax} dx = \delta(a) \qquad \text{and} \qquad \int_{\mathbb{R}} e^{\pi i\, bx^2} dx = \frac{1 + i\,\mathrm{sgn} b}{\sqrt{|b|}}$$

we get the final formula

$$\chi_{p,q}(g(a_1, a_2, a_3, a_4)) = \frac{1 + i\,\mathrm{sgn}\,(pa_2)}{|pa_2|^{-\frac{1}{2}}} e^{\pi i(2pa_4 - \frac{a_3^2 p}{a_2} + \frac{qa_2}{p})} \delta(a_1). \tag{36}$$

Remark 4. It is worthwhile to mention that the same result can be obtained in the pure formal way. Namely, we can consider the representation operator (19) as an integral operator with distributional kernel and compute its trace by integrating this kernel along the diagonal.[6]

This principle works in many other cases, in particular, for all unirreps of nilpotent Lie groups and for the representations of principal series of $SL(2, \mathbb{R})$. It would be useful to have a general theorem of this kind which is an infinite-dimensional analogue of the Lefschetz fixed point formula. ♡

[6] Comparing the two approaches we have to keep in mind the equality $g(a_1, a_2, a_3, a_4) = \exp\left(\sum_{i=2}^4 a_i X_i\right) \exp(a_1 X_1)$.

3.7. Infinitesimal characters.

Consider the same group G as a basic example. We know from Lemma 2 that the algebra $Pol(\mathfrak{g}^*)^G$ is generated by two polynomials $P(F) = X_4$ and $Q(F) = 2X_2X_4 - X_3^2$.

According to the general rule, we define for any element $A \in Z(\mathfrak{g})$ the polynomial $p_A \in Pol(\mathfrak{g}^*)^G$ related to A by the formula

$$A = \mathbf{sym}\big(p_A(2\pi i X_1, \ldots, 2\pi i X_n)\big).$$

Since in our case generators X_i, $2 \le i \le 4$, commute, we can omit **sym** in this expression and obtain the following basic elements in $Z(\mathfrak{g})$:

$$A_1 = \frac{1}{2\pi i} X_4, \qquad A_2 = \frac{1}{4\pi^2}(X_3^2 - 2X_2X_4).$$

In Section 3.2 we computed the images of these elements for all unirreps (see Table 1). The results are given in the table:

	$\pi_{p,q}$	$\pi_{0,q}^{\pm}$	$\pi_{0,0}^{c_1,c_2}$
A_1	p	0	0
A_2	q	q	0

Compare this table with the values of the invariant polynomials $P = p_{A_1}$ and $Q = p_{A_2}$ on the coadjoint orbits:

	$\Omega_{p,q}$	$\Omega_{0,q}^{\pm}$	$\Omega_{0,0}^{c_1,c_2}$
P	p	0	0
Q	q	q	0

We see that Rule 7 works perfectly.

Note also that the infinitesimal characters separate generic representations $\pi_{p,q}$ but do not separate $\pi_{0,q}^{+}$ and $\pi_{0,q}^{-}$ or $\pi_{0,0}^{c_1,c_2}$ for different pairs (c_1, c_2).

3.8. Functional dimension.

The notion of functional dimension often allows us to predict the general form of the answer in many decomposition problems. Here we illustrate it on the example of the regular representation of G. The space $L^2(G,\, dg)$ has functional dimension 4, while the generic unirreps have functional dimension 1 and depend on two parameters.

We conclude that these unirreps must enter with multiplicity ∞^1. This is an analogue of the following well-known fact for compact groups: any unirrep occurs in the regular representation with multiplicity equal to its dimension.

Now let us consider the space $L^2(G,\, dg)$ as the representation space for $\operatorname{Ind}_G^{G\times G} 1$.

Exercise 6. a) Show that the generic unirreps entering in the decomposition of $\operatorname{Ind}_G^{G\times G} 1$ have the form $\pi_{p,q} \times \pi^*_{p,q}$.

b) Prove that $\pi^*_{p,q} \simeq \pi_{-p,q}$.

Hint. Use Rules 4 and 8 of the User's Guide. ♣

Thus, this time the irreducible components have functional dimension 2 and, as before, depend on two parameters. Therefore, we can expect the multiplicity to be finite (it is actually 1 in this case).

This is also the analogue of a well-known fact: for a compact group G all the representations $\pi \times \pi^*$, $\pi \in \widehat{G}$, of $G \times G$ occur with multiplicity 1 in $L^2(G,\, dg)$.

3.9. Plancherel measure.

Let us now compute the Plancherel measure μ on $\widehat{G}$. We shall use the values p, q as local coordinates on the part of $\widehat{G}$ which consists of generic representations. The remaining part has measure zero and can be neglected.

By definition of the Plancherel measure, we have the equality

$$\delta(g) = \int_{\mathbb{R}^2} \chi_{p,q}(g)d\mu(p,q). \tag{37}$$

The direct computation of μ using formula (36) for the generalized character is rather complicated. But Rule 10 of the User's Guide gives the answer immediately:

$$\mu = \frac{1}{2}|dp \wedge dq|. \tag{38}$$

Indeed, from the parametrization above we see that for any integrable function ϕ on $\mathfrak{g}^*$ we have

$$\begin{aligned}\int_{\mathfrak{g}^*} \phi(X_1, X_2, X_3, X_4)d^4X &= \int_{\mathbb{R}^4} \phi\left(x,\, \frac{q+y^2}{2p},\, y,\, p\right) \left| \frac{dx \wedge dq \wedge dy \wedge dp}{2p} \right| \\ &= \int_{\mathbb{R}^2} \frac{|dp \wedge dq|}{2} \left(\int_{\Omega_{p,q}} \phi(F_{x,y}) \cdot \sigma_{p,q} \right).\end{aligned}$$

Applying this formula to $\psi = \widehat{f}$, the Fourier transform of f, we get

$$f(0) = \int_{\mathbb{R}^2} \operatorname{tr}\, \pi_{p,q}(f) \frac{|dp \wedge dq|}{2},$$

which is equivalent to (37) with μ given by (38).

3.10. Other examples.

We highly recommend that the reader (or the lecturer who will use this book) explore independently other examples of nilpotent groups. We list below some of the most interesting and instructive examples.

1. The group G_n of all upper triangular real matrices of order n with unit diagonal. The corresponding Lie algebra $\mathfrak{g}_n$ consists of strictly upper triangular real matrices (those with zero diagonal). The classification of coadjoint orbits in this case is still unknown and is related to deep combinatorial problems (see [**Ki10**] for details).

2. Let $\mathfrak{g}_{n,k}$ be a universal nilpotent Lie algebra of nilpotency class k with n generators. By definition, it is a quotient of a free Lie algebra with n generators by its k-th derivative spanned by all commutators of length $k+1$. The corresponding group $G_{n,k}$ was used by Brown [**Br**] to prove the homeomorphism between $\widehat{G}$ and $\mathcal{O}(G)$ (see Section 3.1).

3. Let $\mathfrak{v}_n$ be the Lie algebra with basis $X_1, \ldots, X_n$ and commutation relations

$$[X_i,\ X_j] = \begin{cases} (j-i)X_{i+j} & \text{if } i+j \le n, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding Lie group V_n can be realized as a group of (equivalence classes of) transformations ϕ of a neighborhood of the origin in the real line that have the form:

$$\phi(x) = x \cdot \left(1 + \sum_{k=1}^{n} a_k x^k\right) + o(x^{n+1}).$$

The group law is the composition of transformations. Note that the 1-parametric subgroups corresponding to basic vectors in $\mathfrak{v}_n$ can be explicitly calculated:

$$\exp(tX_k) : x \mapsto \frac{x}{\sqrt[k]{1-ktx^k}} = x + tx^{k+1} + o(t).$$

The structure of $Z(\mathfrak{v}_n)$ is known. When n is odd, the center has only one generator – the central element $X_n \in \mathfrak{v}$.

When $n = 2m$ is even, there exists another generator $A_m = \mathbf{sym}\,(P_m)$ where $P_m \in Pol^{V_n}(\mathfrak{v}_n^*)$ is a polynomial of degree m and weight $m(2m-1)$. (Here we agree that X_k has weight k.) For small m it can be found using the general scheme of Chapter 1. It turns out that for polynomials P_m there exists a nice generating function.[7]

[7] This result is due to my former student A. Mihailovs.

Namely, let us introduce polynomials $Q_m(y_1, \ldots, y_m)$ by the formula

$$1 + \sum_{m \geq 1} Q_m t^m = \sqrt{1 + \sum_{k \geq 1} y_k t^k}.$$

Then the polynomial P_m can be written as

$$P_m(X_m, \ldots, X_{2m}) = X_{2m}^m \cdot Q_m\left(\frac{X_{2m-1}}{X_{2m}}, \ldots, \frac{X_m}{X_{2m}}\right).$$

4. Generalizing example 1, we can consider a unipotent radical N of a parabolic subgroup P in a semisimple group G. There are a few general results in this case but the detailed analysis of representations has never been done.

A nice particular case is the group $G_n^{\pm}$ of upper triangular matrices that are either symmetric or antisymmetric with respect to the second diagonal. The study of their orbits and representations along the scheme of [**Ki10**] is a very interesting and difficult problem.

5. Among two-step nilpotent groups (i.e. groups of type $\mathbb{R}^m \ltimes \mathbb{R}^n$) the most interesting are the so-called **groups of Heisenberg type**. They are defined as follows.[8]

Let A be one of the following: the complex field $\mathbb{C}$, the skew field $\mathbb{H}$ of quaternions, or the non-associative division algebra $\mathbb{O}$ of Cayley numbers (octonions).

Denote by A_0 the subspace of pure imaginary elements of A. In the space $\mathfrak{g}_{k,l}(A) := A^k \oplus A^l \oplus A_0$ we define a Lie algebra structure by

$$\big[(X', Y', Z'), (X'', Y'', Z'')\big] = (X' + X'', Y' + Y'', Z' + Z'' + \Im(X'\overline{X''} + Y''\overline{Y'}))$$

where $X'\overline{X''} = \sum_{i=1}^k X_i'\overline{X_i''}$, $Y''\overline{Y'} = \sum_{j=1}^l Y_j''\overline{Y_j'}$, and $\Im$ denotes the imaginary part of an element of A.

The characteristic property of the Lie algebras $\mathfrak{g}_{k,l}(A)$ is that all coadjoint orbits are either points, or linear affine manifolds of the same dimension $(k+l)\dim A$. The case $A = \mathbb{C}$ corresponds to the generalized Heisenberg algebra $\mathfrak{h}_{k+l}$.

The corresponding Lie groups $G_{k,l}(A)$ were used in [**GW**] and [**GWW**] to construct non-diffeomorphic compact Riemannian manifolds with the same spectrum of the Laplace-Beltrami operator.

[8]This construction is taken from A. Kaplan, Trans. Amer. Math. Soc. **258** (1980), no. 1, 147–153.

4. Proofs

The main merit of the Orbit Method is that it provides the right geometric formulation of representation-theoretic facts. When this formulation is found, the proofs of the results mentioned in the User's Guide become rather natural and simple, although sometimes we have to use some deep facts from functional analysis and Lie theory.

For nilpotent groups most of the proofs use induction on the dimension of the group.

4.1. Nilpotent groups with 1-dimensional center.

Nilpotent Lie groups have a very nice property (which they share with exponential groups, see below): the exponential map is one-to-one. This map establishes the bijection between Lie subalgebras in $\mathfrak{g} = \mathrm{Lie}(G)$ and connected Lie subgroups in G, and between ideals in $\mathfrak{g}$ and connected normal subgroups in G.

Lemma 4. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then any subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ of codimension 1 is an ideal.*

Proof. From the definition of a nilpotent Lie algebra it follows that the adjoint action of $\mathfrak{g}$ is given by nilpotent operators.[9] If $X \in \mathfrak{g}_0$, the operator $\operatorname{ad} X$ is nilpotent on $\mathfrak{g}$ and preserves $\mathfrak{g}_0$. Hence, it defines a nilpotent operator in the quotient space $\mathfrak{g}/\mathfrak{g}_0$. But this space is 1-dimensional and the corresponding operator must be zero. Hence, $[X, Y] \in \mathfrak{g}_0$ for any $Y \in \mathfrak{g}$ and $\mathfrak{g}_0$ is an ideal. □

Another property of a nilpotent Lie algebra $\mathfrak{g}$ which follows from the definition is that $\mathfrak{g}$ has a non-zero center $\mathfrak{z}$. In the induction procedure below the important role is played by nilpotent Lie algebras with 1-dimensional center. Here we describe the structure of such algebras.

Lemma 5. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with a 1-dimensional center $\mathfrak{z}$. Then there exists a basis $\{X, Y, Z, W_1, \ldots, W_k\}$ in $\mathfrak{g}$ with the following properties:*

$$\mathfrak{z} = \mathbb{R} \cdot Z, \qquad [X, Y] = Z, \qquad [W_i, Y] = 0, \ 1 \le i \le k.$$

Proof. Let Z be any non-zero element of $\mathfrak{z}$. The algebra $\widetilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{z}$ is nilpotent, hence it has a non-zero center $\widetilde{\mathfrak{z}}$. Choose $Y \in \mathfrak{g}$ so that $\widetilde{Y} = Y \mod \mathfrak{z}$ is a non-zero element of $\widetilde{\mathfrak{z}}$.

[9]In fact, this property is characteristic. By Engel's theorem, if $\operatorname{ad} X$ is nilpotent for any $X \in \mathfrak{g}$, then $\mathfrak{g}$ is a nilpotent Lie algebra.

Since $\widetilde{Y} \neq 0$, Y is a non-central element of $\mathfrak{g}$. Hence, there exists $X \in \mathfrak{g}$ such that $[X, Y] \neq 0$. But $\widetilde{Y}$ is central in $\widetilde{\mathfrak{g}}$, hence Y is central modulo $\mathfrak{z}$. We conclude that $[X, Y] \in \mathfrak{z}$. So, replacing X by cX if necessary we can assume that $[X, Y] = Z$.

Now, let $\mathfrak{g}_0$ denote the centralizer of Y in $\mathfrak{g}$. Again using the fact that Y is central modulo $\mathfrak{z}$, we see that $\mathfrak{g}_0$ has codimension 1 in $\mathfrak{g}$. Let $\{W_1, \ldots, W_k\}$ be elements of $\mathfrak{g}_0$ which together with Y and Z form a basis in $\mathfrak{g}_0$. Then all equations above are satisfied. $\square$

In the rest of this section we fix the following notation:

$G_0 \subset G$ – the Lie subgroup corresponding to the Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$; it is the centralizer of an element Y under the adjoint action.

$A \subset G_0$ – the normal abelian 2-dimensional subgroup in G corresponding to the Lie subalgebra $\mathfrak{a}$ spanned by Y and Z.

$C = \exp \mathfrak{z}$ – the central subgroup of G.

Theorem 2. *Let (π, H) be a unirrep of G. Then one of the following must occur:*

1) *the representation π is trivial on C, that is, $\pi(c) = 1_H$ for all $c \in C$;*

or

2) *the representation π has the form $\mathrm{Ind}_{G_0}^{G}\rho$ where ρ is a unirrep of G_0, non-trivial on C.*

Proof. We start by introducing the structure of a G-manifold M on the real plane. Namely, we put $M = \widehat{A}$, the Pontryagin dual to the abelian normal subgroup $A = \exp \mathfrak{a} \subset G$.

The group G acts on A by inner automorphisms, hence it acts on $M = \widehat{A}$. We shall describe this action in more detail.

Introduce the coordinates (y, z) in the vector space $\mathfrak{a}$ with respect to the basis (Y, Z) and transfer these coordinates to A via the exponential map. So, the point $\exp(yY + zZ) \in A$ has coordinates (y, z).

The dual group $\widehat{A}$ consists by definition of multiplicative characters of A that have the form

$$\chi_{\mu,\lambda}(y, z) = e^{2\pi i(\mu y + \lambda z)}.$$

So, $M = \widehat{A}$ is a 2-dimensional plane with coordinates (μ, λ).

Lemma 6. *Let $g \in G$ have the form $g = g_0 \exp tX$, $g_0 \in G_0$, $t \in \mathbb{R}$. Then the action of g on M is*

$$K(g): \ (\mu, \lambda) \mapsto (\mu - t\lambda, \lambda). \tag{39}$$

Proof. First, we note that the subgroup G_0 acts trivially on A, hence on M. It remains to compute the action of the 1-parametric subgroup $\exp tX$, $x \in \mathbb{R}$ on A and on $\widehat{A}$. We have

$$\mathrm{Ad}(\exp tX)(yY + zZ) = \exp(\mathrm{ad}\, tX)(yY + zZ) = yY + (z + ty)Z.$$

The coadjoint action $K(g)$ on $\widehat{A}$ is the dual one, hence given by (39). □

Lemma 7. *The action of G on M is tame.*

Proof. The G-orbits in M are lines $\lambda = const \neq 0$ and points $(\mu, 0)$. Consider the following family of G-invariant sets:

the stripes $S_{a,b} : a < \lambda < b$ and the intervals $I_{a,b} : \lambda = 0, \quad a < \mu < b$ with rational a and b.

This family is countable and separates the orbits. Therefore, the action is tame. □

The next step is the construction of a representation Π of M compatible with a given unitary representation (π, H). Let $f \in \mathcal{A}(M)$. Denote by $\widehat{f}$ the function on A given by

$$\widehat{f}(y, z) = \int_M f(\mu, \lambda)\chi_{\mu,\lambda}(y, z)\, d\mu \wedge d\lambda.$$

Then we put

$$\Pi(f) = \pi(\widehat{f}) = \int_A \widehat{f}(y, z)\pi(\exp(yY + zZ))\, dy \wedge dz.$$

The equalities

$$\Pi(f_1 f_2) = \pi(\widehat{f_1 f_2}) = \pi(\widehat{f_1} * \widehat{f_2}) = \pi(\widehat{f_1})\pi(\widehat{f_2}) = \Pi(f_1)\Pi(f_2),$$
$$\Pi(\overline{f}) = \pi(\widehat{\overline{f}}) = \pi(\widehat{f})^\vee = \pi(f)^*$$

show that (Π, H) is a representation of M. The compatibility condition follows from the definition of the action of G on M.

Suppose now that π is irreducible. According to Theorem 10 in Appendix V.2.4, the representation Π of M is in fact the representation of some G-orbit $\Omega \subset M$. If this orbit is a point $(\mu, 0) \in M$, π is trivial on C. If Ω is a line $\lambda = const \neq 0$, π is induced from a stabilizer of some point $(\mu, \lambda) \in \Omega$. This stabilizer actually does not depend on μ and is exactly G_0. □

The real impact of Theorem 2 is that in both cases of the alternative the study of a unirrep π of G can be reduced to the study of some unirrep of a smaller group: either the quotient group G/C or the subgroup $G_0 \subset G$.

4.2. The main induction procedure.

In this section we give the proofs of most of the results mentioned in the "User's Guide". We shall do it by induction in the dimension of G. It is a sometimes tedious but straightforward way. Of course, more conceptual proofs would be nicer but as of now they are known only for some of the results.

Base of induction. When $\dim G = 1$, we have $G \simeq \mathbb{R}$ and all statements can be easily verified. Note, however, that in the course of this verification we have to use the following classical but quite non-trivial fact.

Proposition 5. *The Pontryagin dual to the group $\mathbb{R}$ is isomorphic to $\mathbb{R}$ itself.*

In other words, all unirreps of $\mathbb{R}$ are 1-dimensional and have the form

$$\pi_\lambda(x) = e^{2\pi i\lambda x}, \quad \lambda \in \mathbb{R}. \tag{40}$$

For the sake of the interested reader we recall the short proof of this fact.

We know that any family of commuting unitary operators in a Hilbert space can be simultaneously diagonalized (see Appendix IV.2.4). Therefore, a unirrep of any abelian group must be 1-dimensional.

Thus, we have to solve the following functional equation that expresses the multiplicativity property

$$\chi(t+s) = \chi(t) \cdot \chi(s).$$

There are two ways to do it.

First, we can use Proposition 3 from Appendix V.1.2, which claims that all characters are smooth functions. From the functional equation above we derive the differential equation $\chi'(t) = a \cdot \chi(t)$ with $a = \chi'(0)$. The general solution has the form $\chi(t) = ce^{at}$. Since $|\chi| \equiv 1$ and $\chi(0) = 1$, the constant a is pure imaginary and $c = 1$. Hence, $\chi(t)$ has the form (40).

Second, if we do not know *a priori* that χ is smooth, we can consider it as a generalized function and use the fact that all generalized solutions to the above equation are in fact ordinary smooth functions. Technically it is more convenient to deal with $\log \chi$ instead of χ itself. □

Now we are in a position to prove the main results by induction on the dimension of the nilpotent Lie group G in question.

Induction Theorem. *Assume that Rules 1–10 are true for all connected and simply connected nilpotent Lie groups of dimension $< n$. Then they are valid also for a connected and simply connected nilpotent Lie group G of dimension n.*

Proof. Let $\mathfrak{g} = \mathrm{Lie}(G)$, and let $\mathfrak{z}$ denote the center of $\mathfrak{g}$. First of all we separate two cases:

I. $\dim \mathfrak{z} > 1$;

II. $\dim \mathfrak{z} = 1$.

In the first case the group G has no faithful unirreps. Indeed, if $\dim \mathfrak{z} = k > 1$ and $Z_1, \ldots, Z_k$ is a basis in $\mathfrak{z}$, then for any unirrep π of G the operators $\pi\left(\exp \sum_{j=1}^{k} x_i Z_j\right)$ are scalar, hence have the form

$$\pi\left(\exp \sum_{i=1}^{k} x_j Z_j\right) = e^{2\pi i \sum_j \lambda_j x_j}.$$

Let $\mathfrak{z}_0 = \{\sum_j x_j Z_j \mid \sum_j \lambda_j x_j = 0\}$. Then $C_0 = \exp \mathfrak{z}_0$ is in the kernel of π.

Therefore, we can consider π as the composition $\widetilde{\pi} \circ p$ where $p : G \to \widetilde{G} = G/C_0$ is the projection on the quotient group and $\widetilde{\pi}$ is some unirrep of $\widetilde{G}$.

The next step is verification of Rules 1–10 for the group G under the assumption that they are valid for $\widetilde{G}$. For some of them, namely for Rules 2–4 and 6–9, this verification is almost evident.

Consider, for example, Rule 2. We know by assumption that this rule is true for the group $\widetilde{G}$. So, any unirrep $\widetilde{\pi}$ of $\widetilde{G}$ has the form

$$\widetilde{\pi} = \pi_{\widetilde{\Omega}} = \mathrm{Ind}_{\widetilde{H}}^{\widetilde{G}} \rho_{\widetilde{F},\widetilde{H}}$$

for some orbit $\widetilde{\Omega} \subset \widetilde{\mathfrak{g}}^*$ and a point $\widetilde{F} \in \widetilde{\Omega}$.

It remains to verify that for $H := p^{-1}(\widetilde{H})$ we have

$$\pi := \widetilde{\pi} \circ p = \mathrm{Ind}_H^G \rho_{F,H}$$

where $\Omega = p^*(\widetilde{\Omega})$ and $F \in p^*(\widetilde{\Omega})$. (Here we denote by p^* the canonical injection of $\widetilde{\mathfrak{g}}^*$ into $\mathfrak{g}^*$ dual to the projection $p : \mathfrak{g} \to \widetilde{\mathfrak{g}}$.) This verification is an easy exercise on the definition of an induced representation and we leave it to the interested (or sceptical) reader.

The same scheme works for the other rules except Rules 1, 5, and 10.

Rule 5 for G actually follows from Rule 3 for $G \times G$ since the tensor product can be written in terms of a direct product:

$$\pi_1 \otimes \pi_2 = \mathrm{Res}_G^{G \times G}(\pi_1 \times \pi_2).$$

For direct products all the rules follow immediately from the same rules for the factors.

So we have to check Rules 1 and 10. Their common feature is that these rules deal not with an individual representation but with the total set $\widehat{G}$. Let Z be the center of G.

To prove that $\widehat{G} = \mathcal{O}(G)$ as a set, it is enough

(i) to split the set $\widehat{G}$ of all unirreps according to their restrictions to Z;

(ii) to split the set $\mathcal{O}(G)$ of coadjoint orbits according to their projection to $\mathfrak{z}^*$; and

(iii) establish bijections between corresponding parts.

This is also an easy exercise and can be left for the reader. But to prove that the final bijection is a homeomorphism is a more difficult problem. We shall discuss it later.

Finally, Rule 10 follows from Rule 6 which gives the explicit formula for the generalized character. Indeed, if we consider characters as distributions, the Plancherel formula gives the decomposition of the δ-function supported at $e \in G$ into distributional characters of unirreps according to (37) (see also (10) in Section 2.7). After the Fourier transform in the exponential coordinates and using Rule 6 this formula becomes

$$(41)\qquad \int_{\mathfrak{g}^*} f(F)dF = \int_{\mathcal{O}(G)} \left(d\mu(\Omega) \int_{\Omega} f(F) vol_{\Omega}(F) \right).$$

This gives exactly Rule 10 for G.

Consider now case II. Here we have the alternative of Theorem 2. For a representation of the first kind, Rules 2–6 and 8–9 are verified exactly as in case I. For representations of the second kind, this is also true but for another reason. As an illustration we give here the proof of Rule 6.

We keep the notation of Theorem 2. Let $\pi = \mathrm{Ind}\,_{G_0}^{G}\rho$ be a unirrep of G. Let ρ act in a Hilbert space V. Then π can be realized in $\mathcal{H} = L^2(\mathbb{R}, V, dx)$ by the formula

$$(42)\qquad (\pi(g_0 \exp tX)f)(x) = \rho(\exp xXg_0(\exp xX)^{-1})f(x+t).$$

In the next section we shall prove the existence of the generalized character and show that it can be computed by the standard formula using the distributional kernel of $\pi(g)$. Here we perform the corresponding computation.

From (42) we obtain the formula for the distributional kernel of π in terms of the distributional kernel of ρ:

$$K_\pi(g_0 \exp(tX) \,|\, x, y) = K_\rho(\exp xXg_0(\exp xX)^{-1})\delta(x+t-y).$$

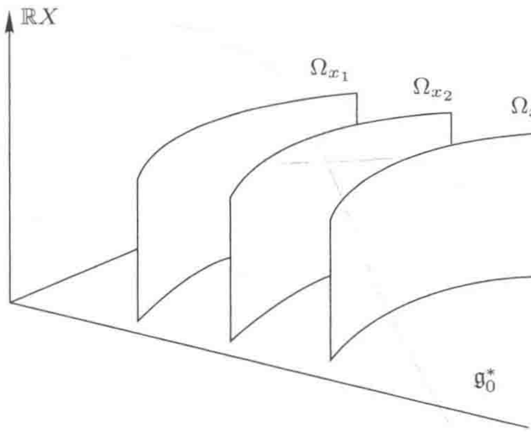

Figure 1

It follows that

$$\chi_\pi(g_0 \exp tX) = \delta(t) \cdot \int_{\mathbb{R}} \chi_\rho(\exp xX g_0 (\exp xX)^{-1}) dx, \tag{43}$$

which is an exact analog of the Frobenius formula for the character of induced representation of a finite group (see Appendix V.2.1).

Now we use Rule 6 for the group G_0 and write

$$\chi_\rho(\exp X_0) = \int_{\Omega_0} e^{2\pi i \langle F_0,\, X_0\rangle} vol_{\Omega_0}(F_0).$$

This together with (43) gives us the formula

$$\begin{aligned} \chi_\pi(\exp X_0 \exp tX) &= \delta(t) \int_{\mathbb{R}} \left(dx \int_{\Omega_0} e^{2\pi i \langle K(\exp xX)F_0,\, X_0\rangle} vol_{\Omega_0}(F_0) \right) \\ &= \delta(t) \int_{\mathbb{R}} \left(dx \int_{\Omega_x} e^{2\pi i \langle F_x,\, X_0\rangle} vol_{\Omega_x}(F_x) \right) \end{aligned} \tag{44}$$

where we use the notation $\Omega_x := K(\exp xX)\Omega_0$ and denote by F_x a generic element of Ω_x.

On the other hand, Rule 6 for π looks like

$$\chi_\pi(\exp(X_0 + tX)) = \int_\Omega e^{2\pi i \langle F,\, X_0 + tX\rangle} vol_\Omega(F). \tag{45}$$

But the orbit Ω is a cylinder with $\bigcup_{x\in\mathbb{R}} \Omega_x$ as a base and $\mathbb{R}\cdot X$ as a directrix (see Figure 1).

The generic point of Ω has the form $F = F_x + sF_1$ where $F_x \in \Omega_x$, $s \in \mathbb{R}$, $\langle F_1,\, X\rangle = 1$, and $\langle F_1,\, X_0\rangle = 0$ for $X_0 \in \mathfrak{g}_0$.

Therefore the integral over Ω in (45) takes the form

$$\begin{aligned}\chi_\pi(\exp(X_0+tX)) &= \int_{\mathbb{R}^2}\left(dxds\int_{\Omega_x} e^{2\pi i\langle F_x+sF_1,\, X_0+tX\rangle} vol_{\Omega_x}(F_x)\right)\\ &= \int_{\mathbb{R}^2}\left(dxds\int_{\Omega_x} e^{2\pi i(\langle F_x,\, X_0\rangle+st)} vol_{\Omega_0}(F_0)\right)\\ &= \delta(t)\int_{\mathbb{R}}\left(dx\int_{\Omega_x} e^{2\pi i\langle F_x,\, X_0\rangle} vol_{\Omega_x}(F_x)\right).\end{aligned}$$

The last expression coincides with (44) and proves Rule 6 for G.

We turn now to Rule 7. The difficulty with this rule is that the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ changes when we pass from $\mathfrak{g}$ to a subalgebra $\mathfrak{g}_0$ or to a quotient algebra $\widetilde{\mathfrak{g}}$. Fortunately, this change can be controlled.

It is convenient to identify $Z(\mathfrak{g})$ with $Pol(\mathfrak{g}^*)^G$, $Z(\mathfrak{g}_0)$ with $Pol(\mathfrak{g}_0)^{G_0}$, and $Z(\widetilde{\mathfrak{g}}^*)$ with $Pol(\widetilde{\mathfrak{g}})^{\widetilde{G}}$.

Consider the case when we pass from $\mathfrak{g}$ to a quotient algebra $\widetilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{c}$ where $\mathfrak{c}$ is a subalgebra of $\mathfrak{z}$. We denote by p the projection of $\mathfrak{g}$ onto $\widetilde{\mathfrak{g}}$ (and also the projection of G to $\widetilde{G}$). Then the dual map p^* identifies $\widetilde{\mathfrak{g}}^*$ with the subspace $\mathfrak{c}^\perp \subset \widetilde{\mathfrak{g}}^*$. Let Ω be a G-orbit in this subset.

We denote by $p' : S(\mathfrak{g}) \cong Pol(\mathfrak{g}^*) \to S(\widetilde{\mathfrak{g}}) \cong Pol(\widetilde{\mathfrak{g}}^*)$ the projection of algebras that corresponds to the projection p of sets. For any element $A \in Z(\mathfrak{g}) \cong Pol(\mathfrak{g}^*)^G$ we have $p(A) \in Z(\widetilde{\mathfrak{g}})$. Hence, Rule 7 is valid for $p(A)$: under the unirrep $\widetilde{T}_\Omega$ of $\widetilde{G}$ it goes to the scalar $P_{p(A)}(\Omega)$. But this means that under the unirrep $T_\Omega := \widetilde{T}_\Omega \circ p$ the element A goes to $P_A(\Omega)$.

Suppose now that the second case of the alternative of Theorem 2 holds. From the commutation relations of Lemma 5 it follows that G-invariant polynomials on $\mathfrak{g}^*$ do not depend on the variable X, hence can be considered as polynomials on $\mathfrak{g}_0^*$. Moreover, $Z(\mathfrak{g}) = Z(\mathfrak{g}_0)^{\exp \mathbb{R}\cdot X}$.

Let π be a unirrep of G induced by a unirrep ρ of G_0. We assume that ρ corresponds to some orbit $\Omega_0 \subset \mathfrak{g}_0^*$, and we denote by Ω the orbit corresponding to π.

Formula (42) shows that any $A \in Z(\mathfrak{g})$ goes under π to a scalar operator with the same eigenvalue as $\rho(A)$. But the former coincide by Rule 7 with $P_A(\Omega_0)$. We can consider $P_A \in Pol(\mathfrak{g}_0^*)$ as a polynomial on $\mathfrak{g}^*$ which does not depend on the X-coordinate. Moreover, this polynomial is invariant under the action of $\exp \mathbb{R}\cdot X$. Hence, it takes the same value on Ω as on Ω_0 (cf. Figure 1 above). $\square$

4.3. The image of $U(\mathfrak{g})$ and the functional dimension.

The main result of this section is

Theorem 3. *Let G be a connected nilpotent Lie group, and let π be a unirrep of G corresponding to a $2n$-dimensional orbit $\Omega \subset \mathfrak{g}^*$. Then π can be realized in the Hilbert space $H = L^2(\mathbb{R}^n,\, d^n x)$ so that*

a) *H^∞ coincides with the Schwartz space $\mathcal{S}(\mathbb{R}^n)$;*

b) *the image of $U(\mathfrak{g})$ under π coincides with the algebra W_n of all differential operators with polynomial coefficients.*

Proof. If the representation π is not locally faithful (i.e. has a kernel of positive dimension), the statement of the theorem reduces to the analogous statement for a group of lower dimension. So, the crucial case is that of a representation (π, H) of a Lie group G with 1-dimensional center for which we have the second alternative of Theorem 2. We keep the notation of this theorem and its proof. Without loss of generality we can assume, using (38), that the inducing representation ρ has the property $\rho_*(Y) = 0$.

Then from (42) we conclude that the images of basic elements of $\mathfrak{g}$ under π_* are:

$$\pi_*(X) = \frac{d}{dx}, \qquad \pi_*(Y) = 2\pi i\lambda x, \qquad \pi_*(Z) = 2\pi i\lambda,$$
$$\pi_*(W_j) = \rho_*(\mathrm{Ad}(\exp xX)W_j).$$

Since the action of $\operatorname{ad} X$ in $\mathfrak{g}$ is nilpotent, the elements

$$\widetilde{W_j}(x) := \mathrm{Ad}(\exp xX)W_j = \exp(\operatorname{ad} xX)W_j$$

can be written as linear combinations of Y, Z, $W_1, \ldots, W_k$ with coefficients which are polynomials in x.

Note that the constant term of $\widetilde{W_j}(x)$ is $\widetilde{W_j}(0) = W_j$.

Now we use the induction hypothesis and choose the realization of V in the form of $L^2(\mathbb{R}^m, d^m y)$ so that the operators $\rho_*(W_j)$ generate the algebra of all differential operators with polynomial coefficients in variables $y_1, \ldots, y_m$.

We see that in the chosen realization the image of $U(\mathfrak{g})$ is contained in the algebra of differential operators with polynomial coefficients in variables x, $y_1, \ldots, y_m$. Moreover, this image contains the operators x and ∂_x. It follows that together with $\rho_*(\widetilde{W_j})$ the image contains $\rho_*(W_j)$. Indeed, the constant term of any polynomial P in x can be expressed as a linear combination of P, $x\partial_x P, \ldots, (x\partial_x)^{\deg P} P$.

We conclude that $\pi(U(\mathfrak{g}))$ contains all differential operators with polynomial coefficients.

We proved the second statement of the theorem. The first follows from it because for any generalized function on $\mathbb{R}^n$ the conditions

$$D\phi \in L^2(\mathbb{R}^n, d^n x) \quad \text{for any } D \in A_n \text{ and } \phi \in \mathcal{S}(\mathbb{R}^n)$$

are equivalent. □

4.4. The existence of generalized characters.

In this section we prove the existence of generalized characters for all unirreps π and show that they can be computed directly in terms of distributional kernels of the operators $\pi(g)$, $g \in G$.

We deduce this from Theorem 3.

For our goal we need an explicit example of a differential operator D with polynomial coefficients in $L^2(\mathbb{R}^n, d^n x)$ for which D^{-1} exists and belongs to the trace class.

We start with $n = 1$ and consider the operator $H = -\frac{d^2}{dx^2} + x^2$. We already considered this operator in Chapter 2. Here we recall briefly the facts needed.

Lemma 8. *The spectrum of H is simple and consists of all odd positive integers* $1, 3, 5, \ldots$.

Proof. Recall the definition of the annihilation and creation operators:

$$a = x + \frac{d}{dx}, \qquad a^* = x - \frac{d}{dx}.$$

In Chapter 2 we derived the relations

$$(46) \quad H = a^*a + 1 = aa^* - 1, \qquad Ha = a(H-2), \qquad Ha^* = a^*(H+2).$$

From the last two relations in (46) we conclude that if a function f is an eigenfunction for H with an eigenvalue λ, then af and a^*f are also eigenfunctions with eigenvalues $\lambda - 2$ and $\lambda + 2$, respectively (only if they are non-zero).

Further, the equations

$$af = 0, \qquad a^*f = 0$$

have general solutions $f = ce^{-\frac{x^2}{2}}$ and $f = ce^{\frac{x^2}{2}}$, respectively.

It follows that the creation operator a^* has the zero kernel in $L^2(\mathbb{R}, dx)$ and the annihilation operator a has the 1-dimensional kernel spanned by $f_0(x) = e^{-\frac{x^2}{2}}$.

From the first relation in (46) we see that $Hf_0 = f_0$. Hence, if we put $f_n := (a^*)^n f_0$, then $Hf_n = (2n+1)f_n$.

To complete the calculation of the spectrum, it remains to show that the linear span V of $f_0, , f_1, \ldots$ is dense in $L^2(\mathbb{R}, dx)$, so that H has the point spectrum described in the lemma. This was done in Chapter 2. □

We see that the operator H has a bounded compact inverse H^{-1} that belongs to the Hilbert-Schmidt class $\mathcal{L}(L^2(\mathbb{R}, dx))$. Therefore H^{-2} is of trace class.

In the space $L^2(\mathbb{R}^n, d^n x)$ we define the operator H_n by

$$H_n = \sum_{k=1}^{n} (-\partial_k^2 + x_k^2).$$

The eigenfunctions and eigenvalues of H_n have the form

$$\psi_k(x_1, \ldots, x_n) = \prod_{i=1}^{n} \psi_{k_i}(x_i), \qquad \lambda_k = \sum_{i=1}^{n} (2k_i + 1)$$

where $k = (k_1, \ldots, k_n)$ is a multi-index and the ψ_k are eigenfunctions of H.

Then an elementary estimation shows that H_n^{-n-1} is of trace class.

Now we are ready to prove

Theorem 4. *Let G be a connected nilpotent Lie group, and let (π, H) be a unirrep of G. Define the Schwartz space $\mathcal{S}(G)$ using the diffeomorphism* $\exp: \mathbb{R}^n \cong \mathfrak{g} \to G$.

Then for any $\phi \in \mathcal{S}(G)$ the operator $\pi(\phi)$ is of trace class in H. (In other words, the character χ_π is a tempered distribution on G.)

Proof. One way to prove the theorem is to show, using Theorem 3, that H can be realized as $L^2(\mathbb{R}^n, d^n x)$ so that $\pi(\phi)$ is an integral operator with kernel in $\mathcal{S}(R^{2n})$. We leave this to the reader.

Another way is more direct. Choose an element $A \in U(\mathfrak{g})$ such that π_* sends it to the operator B with the inverse B^{-1} of trace class (e.g., to the operator H_n^{n+1} considered above). Then for $\phi \in \mathcal{S}(G)$ we can write

$$\pi(\phi) = B^{-1} B \pi(\phi) = B^{-1} \pi_*(A) \pi(\phi) = B^{-1} \pi(A\phi).$$

In the last term the first factor is an operator of trace class while the second is a bounded operator. So, $\pi(\phi)$ is of trace class. Moreover, the trace norm of $\pi(\phi)$ can be estimated:

$$\begin{aligned} \|\pi(\phi)\|_1 &= \max_{\|C\| \leq 1} |\mathrm{tr}\,(\pi(\phi)C)| = \max_{\|C\| \leq 1} |\mathrm{tr}\,(B^{-1}\pi(A\phi)C)| \\ &\leq \mathrm{tr}\, B^{-1} \, \|\pi(A\phi)\|_H \leq \mathrm{tr}\, B^{-1} \, \|A\phi\|_{L^1(G)}. \end{aligned} \tag{47}$$

□

4.5. Homeomorphism of $\widehat{G}$ and $\mathcal{O}(G)$.

Let $\widetilde{G}$ denote the set of all equivalence classes of unitary (not necessarily irreducible) representations of a topological group G. For brevity we usually use the same notation for representations and their equivalence classes.

Recall the definition of the topology on $\widetilde{G}$.

Definition 3. A neighborhood of a given representation (π, H) is determined by the following data:
(i) a compact subset $K \subset G$;
(ii) a positive number ϵ;
(iii) a finite collection of vectors $X = \{x_1, \ldots, x_n\}$ in H.

The neighborhood $U_{K,\epsilon,X}(\pi)$ of (π, H) consists of all unitary representations (ρ, V) such that there exists a finite family $Y = \{y_1, \ldots, y_n\}$ of vectors in V satisfying

$$|(\pi(g)x_i, x_j)_H - (\rho(g)y_i, y_j)_V| < \epsilon \quad \text{for any } g \in K. \tag{48}$$

The set $\widehat{G}$ of unirreps is a subset of $\widetilde{G}$ and inherits the topology from there.

Recall one more notion from the general representation theory. We say that a unirrep π is **weakly contained** in a unitary representation ρ if the point $\pi \in \widehat{G} \subset \widetilde{G}$ is contained in the closure of the point $\rho \in \widetilde{G}$.

Exercise 7. The 1-dimensional representation $\pi_\lambda(t) = e^{2\pi i\lambda t}$ of $\mathbb{R}$ is weakly contained in the regular representation ρ of $\mathbb{R}$ acting by translations in $L^2(\mathbb{R}, dx)$:

$$\big(\rho(t)f\big)(x) = f(x+t).$$

Hint. Make the Fourier transform and consider a δ-like sequence in the dual space $L^2(\mathbb{R}^*, d\lambda)$.

Note that π_λ is not a subrepresentation of ρ. ♣

The main problem we discuss in this section splits into two parts:

(i) prove that the map $\mathcal{O}(G) \to \widehat{G}$ is continuous;

(ii) show that the inverse map is continuous.

The first part is easier and goes as follows. Suppose that a sequence of orbits $\{\Omega_n\}$ goes to a limit Ω. By definition of the quotient topology in $\mathcal{O}(G)$ this means that there exists a sequence of functionals $\{F_n\}$ such that $F_n \in \Omega_n$ and $\lim\limits_{n\to\infty} F_n = F \in \Omega$.

Let $\mathfrak{h}_n$ be a subalgebra of maximal dimension subordinated to F_n, and let $H_n = \exp \mathfrak{h}_n$ be the corresponding subgroup of G. Passing to a subsequence

if necessary, we can assume that all $\mathfrak{h}_n$ have the same codimension $2r$ and have a limit $\mathfrak{h}$ in the Grassmannian $G_{2r}(\mathfrak{g})$. It is clear that $\mathfrak{h}$ is subordinate to F. But it can happen that it does not have the maximal possible dimension (since $\operatorname{rk} B_F$ can be less than $\operatorname{rk} B_{F_n} = 2r$).

Using Lemma 9 from Section 5.2 of Chapter 1 we can construct a subalgebra $\widetilde{\mathfrak{h}}$ of maximal dimension subordinate to F so that $\mathfrak{h} \subset \widetilde{\mathfrak{h}}$, hence $\widetilde{H} = \exp \widetilde{\mathfrak{h}} \supset \exp \mathfrak{h} = H$.

Denote by π (resp. $\widetilde{\pi}$) the induced representation $\operatorname{Ind}_H^G \rho_{F,H}$ (resp. the unirrep $\pi_\Omega = \operatorname{Ind}_{\widetilde{H}}^G \rho_{F,\widetilde{H}}$).

We need to show that $\widetilde{\pi}$ is contained in the limit[10]

$$\lim_{n\to\infty} \pi_{\Omega_n} = \lim_{n\to\infty} \operatorname{Ind}_{H_n}^G \rho_{F_n,H_n}.$$

Lemma 9. *We have* $\lim\limits_{n\to\infty} \pi_{\Omega_n}$ *in* $\widehat{G}$ *contains* π_Ω.

Sketch of the proof. Since the sequence $\mathfrak{h}_n$ tends to $\mathfrak{h}$ and all $\mathfrak{h}_n$ have the same dimension, we can identify $X_n = H_n\backslash G$ with the standard space $\mathbb{R}^n$, so that the actions of G on $\mathbb{R}^n$ arising from the identification $\mathbb{R}^n \cong X_n$ have a limit. It is clear that this limit corresponds to the identification $\mathbb{R}^n \cong X = H\backslash G$. The statement of the lemma follows from the explicit formula for the induced representation given by Rule 2 of the User's Guide.□

In the case $\operatorname{rk} B_F = 2r = \operatorname{rk} B_{F_n}$ we have $\widetilde{H} = H$, $\widetilde{\pi} = \pi$ and we obtain the desired relation $\lim\limits_{n\to\infty} \pi_{\Omega_n} = \pi_\Omega$.

Assume now that $\operatorname{rk} B_F < 2r$, hence H is a proper subgroup in $\widetilde{H}$. Then π is a reducible representation.

Lemma 10. *The representation* $\widetilde{\pi}$ *is weakly contained in* π*, hence is contained in* $\lim\limits_{n\to\infty} \pi_{\Omega_n}$.

Sketch of the proof. First, we observe that $\rho_{F,\widetilde{H}}$ is weakly contained in $\operatorname{Ind}_H^{\widetilde{H}} \rho_{F,H}$. This can be proved exactly as in the statement of Exercise 7. Next we use the following general fact, proved by G. M. J. Fell.

Theorem 5 (see [**Fe**]). *The induction functor* Ind_H^G *defines a continuous map from* $\widehat{H}$ *to* $\widehat{G}$. □

[10] Note that because $\widehat{G}$ is not necessarily Hausdorff, the limit of a sequence need not be a point, but is some closed subset in $\widehat{G}$.

Consider now the continuity of the inverse map: $\widehat{G} \to \mathcal{O}(G)$. To give an idea of how to prove this continuity, we describe the approach invented by I. Brown in [**Br**]. It is based on two simple observations:

1. Any nilpotent Lie algebra can be considered as a quotient of $\mathfrak{g}_{n,k}$ (see example 2 in Section 3.10) for appropriate n and k. Consequently, the topological space $\widehat{G}$ is a subspace of $\widehat{G_{n,k}}$ with inherited topology.

2. The Lie algebra $\mathfrak{g}_{n,k}$ has a huge group of automorphisms induced by automorphisms of the free Lie algebra.

The first observation reduces the general problem to the case $\mathfrak{g} = \mathfrak{g}_{n,k}$, while the second allows us to prove the theorem for $\mathfrak{g}_{n,k}$ by induction on k.

Chapter 4

Solvable Lie Groups

1. Exponential Lie groups

The next class of Lie groups where the orbit method works well is the class of exponential Lie groups. Most of the prescriptions of our User's Guide (namely, Rules 1, 3, 4, 5, 8, 9) are still valid in this more general situation.

These results were obtained mainly by the French school (see [**BCD**]); the validity of Rules 3, 4, and 5 was first proved in [**Bu**]. As for the topological isomorphism of $\widehat{G}$ and $\mathcal{O}(G)$, this was established only recently in [**LL**].

Rules 2, 6, 7, and 10 need modifications, which we discuss below.

1.1. Generalities.

A Lie group G is called **exponential** if the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism. So, the exponential coordinates give a single chart covering the whole group. In particular, all exponential groups are connected and simply connected.

A Lie algebra $\mathfrak{g}$ is called **exponential** if the corresponding connected and simply connected Lie group is exponential.

Example 1. Let $E(2)$ denote the Lie group of rigid motions of the Euclidean plane. We denote by G the connected component of this group and by $\widetilde{G}$ its universal covering. All three groups $E(2)$, G, and $\widetilde{G}$ have the same Lie algebra $\mathfrak{g} = \mathfrak{e}(2)$.

The group G consists of rotations and translations. It has a convenient matrix realization by complex matrices of the form

$$g_{\tau,w} = \begin{pmatrix} \tau & w \\ 0 & 1 \end{pmatrix}, \quad \tau,\, w \in \mathbb{C},\ |\tau| = 1.$$

Hence, the Lie algebra $\mathfrak{g}$ can be realized by matrices

$$A_{t,z} = \begin{pmatrix} it & z \\ 0 & 0 \end{pmatrix}, \qquad t \in \mathbb{R},\ z = x + iy \in \mathbb{C}.$$

It admits the natural basis

$$T = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$$

with the commutation relations

$$[T,\, X] = Y, \qquad [T,\, Y] = -X, \qquad [X,\, Y] = 0.$$

The Lie group $\widetilde{G}$ can be realized by 3×3 matrices

$$\widetilde{g}_{t,w} = \begin{pmatrix} e^{it} & 0 & w \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \qquad t \in \mathbb{R},\ w \in \mathbb{C}.$$

We have

$$\exp_G(tT + xX + yY) = \begin{pmatrix} e^{it} & e^{\frac{it}{2}} s(t)(x+iy) \\ 0 & 1 \end{pmatrix}, \qquad s(t) = \frac{\sin(t/2)}{t/2},$$

$$\exp_{\widetilde{G}}(tT + xX + yY) = \begin{pmatrix} e^{it} & 0 & e^{\frac{it}{2}} s(t)(x+iy) \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

We see that elements $g_{\tau,0} \in G$ have infinitely many preimages, while elements $\widetilde{g}_{2\pi n,w}$, $n \in \mathbb{Z}\backslash\{0\}$, $w \neq 0$, are not covered at all by the exponential map. Therefore, G and $\widetilde{G}$ are not exponential Lie groups and $\mathfrak{g}$ is not an exponential Lie algebra. ♢

Actually, the example above is a crucial one, as the following criterion shows.

Proposition 1. *Let G be a connected and simply connected Lie group. Then G is exponential iff its Lie algebra $\mathfrak{g}$ satisfies the following equivalent conditions:*

a) *The operators* $\operatorname{ad} X$, $X \in \mathfrak{g}$, *have no pure imaginary non-zero eigenvalues.*

b) *The Lie algebra $\mathfrak{g}$ has no subalgebra isomorphic to* $\mathfrak{e}(2)$. □

From Proposition 1 it follows that the class of exponential Lie algebras is hereditary: any subquotient (i.e. quotient algebra of a subalgebra) of an exponential Lie algebra is also exponential.

We note also that the class of exponential Lie algebras is situated strictly between nilpotent and solvable Lie algebras.

1.2. Pukanszky condition.

An important correction is needed to extend Rule 2 of the User's Guide to exponential Lie groups. To formulate it we introduce

Definition 1. Let us say that a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, subordinate to $F \in \mathfrak{g}^*$, satisfies the **Pukanszky condition** if

$$p^{-1}(p(F)) \subset \Omega_F, \tag{1}$$

i.e. the fiber over $p(F)$ in $\mathfrak{g}$ lies entirely in a single G-orbit. Note that this fiber can be written as $p^{-1}(p(F)) = F + \mathfrak{h}^{\perp}$.

The **modified Rule 2** differs from the initial one by the additional requirement:

The Lie subalgebra $\mathfrak{h}$ must satisfy the Pukanszky condition.

In fact, the necessity of the condition follows immediately from Rule 4 of the User's Guide. According to this rule, the spectrum of the induced representation $\operatorname{Ind}_H^G U_{F,H}$ consists of representations π_Ω, for which Ω intersects the set $p^{-1}(\Omega)$. So, the spectrum is reduced to a single point iff the Pukanszky condition is satisfied.

Counting the functional dimension by Rule 9 of the User's Guide, we can expect that the multiplicity of the spectrum is finite (in fact, it is equal to 1) when the following equality takes place:

$$\operatorname{codim}_{\mathfrak{g}} \mathfrak{h} = \frac{1}{2} \dim \Omega_F, \tag{2}$$

i.e. when $\mathfrak{h}$ is a maximal isotropic subspace of $\mathfrak{g}$ for the bilinear form B_F.

This is indeed true.

Theorem 1 (Pukanszky). *Conditions* (1) *and* (2) *together are necessary and sufficient for the irreducibility of the representation*

$$\pi_{\Omega_F} = \operatorname{Ind}_H^G U_{F,H}. \tag{3}$$

Proof. The proof follows essentially the same scheme that we used in the previous chapter to prove Rule 2 for nilpotent Lie groups. However it is much more involved. So, we omit it and refer to [**P1**], [**BCD**], or [**Bu**] for the details. □

Theorem 1 holds, in particular, for nilpotent Lie groups. We have not included the Pukanszky condition in the initial Rule 2 because of the following fact.

Proposition 2. *For nilpotent Lie groups the Pukanszky condition* (1) *is satisfied automatically.*

The statement follows from two lemmas.

Lemma 1. *Let $\mathfrak{g}$ be any Lie algebra, and let $\mathfrak{h} \subset \mathfrak{g}$ be any Lie subalgebra that is subordinate to F and has codimension $\frac{1}{2}\dim\Omega_F$ in $\mathfrak{g}$. Then the affine manifold $F + \mathfrak{h}^{\perp}$ has an open intersection with Ω_F (i.e. the local version of the Pukanszky condition is always satisfied).*

Proof. Let $H = \exp \mathfrak{h}$. Consider the H-orbit of F in $\mathfrak{g}^*$. Let us check that this orbit is contained in $F + \mathfrak{h}^{\perp}$. For any $X, Y \in \mathfrak{h}$ and $h = \exp(X)$ we have

$$\langle K(h)F, Y\rangle = \langle F, \mathrm{Ad}(h)^{-1}Y\rangle = \langle F, e^{\mathrm{ad}(-X)}Y\rangle = \langle F, Y\rangle$$

since $e^{\mathrm{ad}(-X)}Y \in Y + [\mathfrak{h}, \mathfrak{h}]$ and $F\big|_{[\mathfrak{h}, \mathfrak{h}]} = 0$. So, F and $K(h)F$ have the same values on $\mathfrak{h}$, hence, the difference is in $\mathfrak{h}^{\perp}$.

Now we compare dimensions of $K(H)F$ and $\mathfrak{h}^{\perp}$. The first dimension is equal to

$$\dim H - \dim(\mathrm{Stab}(F) \cap H) \geq \dim H - \dim \mathrm{Stab}(F) = \frac{1}{2}\dim \Omega.$$

The second is $\dim \mathfrak{g} - \dim \mathfrak{h} = \frac{1}{2}\dim \Omega$.

We see that $\dim K(H)F \geq \dim \mathfrak{h}^{\perp}$. But $K(H)F$ is contained in $F + \mathfrak{h}^{\perp}$. Hence, both sets have the same dimension. It follows that the intersection $\Omega_F \cap (F + \mathfrak{h}^{\perp})$ contains a neighborhood of F in $F + \mathfrak{h}^{\perp}$.

The same argument can be applied to any point $F_1 \in F + \mathfrak{h}^{\perp}$. Hence, the set $\Omega_F \cap (F + \mathfrak{h}^{\perp})$ is open in $F + \mathfrak{h}^{\perp}$. □

Lemma 2. *Let G be a connected unipotent*[1] *Lie subgroup in $GL(n, \mathbb{R})$. Then all G-orbits in $\mathbb{R}^n$ are affine algebraic subvarieties in $\mathbb{R}^n$ (i.e. they are defined by a system of polynomial equations). In particular, they are closed in the ordinary topology of $\mathbb{R}^n$.*

Proof. Induction by n (see details in [**Ki1**] or [**Di1**]). □

Let us return to the proof of Proposition 2. According to Lemma 2, for nilpotent Lie groups all coadjoint orbits are closed. Hence, $K(G)F \cap (F + \mathfrak{h}^{\perp})$ is closed in $(F + \mathfrak{h}^{\perp})$.

[1] A group $G \subset GL(n, \mathbb{R})$ is called **unipotent** if it consists of matrices with all eigenvalues equal to 1. (Equivalent property: G is conjugate to a subgroup of N, the group of triangular matrices with 1's on the main diagonal.)

But according to Lemma 1, it is also open. So, it must coincide with $(F + \mathfrak{h}^{\perp})$. □

For general exponential groups, Lemma 2 is no longer true (see for instance Example 2 below) and we have to include the Pukanszky condition in the formulation of Rule 2.

1.3. Restriction-induction functors.

Rules 3 and 4 of the User's Guide remain true and in some cases admit a more precise formulation. Namely, assume that an orbit $\Omega \subset \mathfrak{g}^*$ intersects the preimage $p^{-1}(\omega)$ along a set with several connected components. Then the multiplicity $m(\Omega, \omega)$ depends on the number of components (see [**BCD**] for details).

1.4. Generalized characters.

Rule 6 of the User's Guide needs corrections for the two reasons discussed below.

First, for non-nilpotent exponential groups the generalized characters of unirreps are not necessarily well defined as distributions. Namely, it could happen that the operator $\pi(\phi)$ does not belong to the trace class even for $\phi \in \mathcal{A}(G)$.

The explanation of this phenomenon by the orbit method is very simple. The coadjoint orbits are no longer closed submanifolds in $\mathfrak{g}^*$. As a consequence, the canonical volume form associated with the symplectic structure on an orbit can have a singularity at the boundary and the integral formula of Rule 6 does not define a distribution on G.

So, we have to restrict the domain of definition of the generalized character by imposing additional conditions on the test functions $\phi \in \mathcal{A}(G)$. In the simplest case (see Example 2 below) the additional condition is very natural: the **Fourier transform**

$$\tilde{\phi}(F) = \int_{\mathfrak{g}} \phi(\exp X) \cdot e^{2\pi i \langle F, X \rangle} dX \tag{4}$$

must vanish on the boundary of the orbit.

In general, a more severe restriction is needed to compensate for the singularity of the canonical measure at the boundary. Note that for closed orbits these additional restrictions are unnecessary.

Second, for nilpotent groups the Fourier transform $\phi \mapsto \tilde{\phi}$ given by (4) defines the unitary transformation from $L^2(G, dg)$ to $L^2(\mathfrak{g}^*, dF)$ where dg is the Haar measure on G and dF is the Lebesgue measure on $\mathfrak{g}^*$, both suitably normalized. In terms of this transform Rule 6 looks like

$$\langle \chi_\Omega, \phi \rangle := \operatorname{tr} T_\Omega(\phi) = \int_\Omega \tilde{\phi}(F) \cdot \frac{\sigma^r}{r!} \quad \text{where } r = \frac{\dim \Omega}{2}. \tag{5}$$

This can be briefly formulated as follows:

The Fourier transform of χ_Ω is the canonical measure on the coadjoint orbit Ω.

In the case of non-nilpotent groups we have to take into account the more complicated relation between invariant measures on G and on $\mathfrak{g}^*$.

For non-unimodular groups the densities of the right and left Haar measures have the following form in canonical coordinates:

$$d_r(\exp X) = \det\left(\frac{e^{\operatorname{ad} X} - 1}{\operatorname{ad} X}\right) dX, \quad d_l(\exp X) = \det\left(\frac{1 - e^{-\operatorname{ad} X}}{\operatorname{ad} X}\right) dX.$$

For unimodular groups both expressions can be written as $q(X)dX$ where

$$q(X) = \det\left(\frac{\sinh(\operatorname{ad} X/2)}{\operatorname{ad} X/2}\right).$$

This is an analytic function on $\mathfrak{g}$ whose zeros coincide with singular points of the exponential map. For exponential groups, $q(X)$ is everywhere positive and so the function

$$(6) \qquad p(X) = \sqrt{q(X)} = \left(\det\left(\frac{\sinh(\operatorname{ad} X/2)}{\operatorname{ad} X/2}\right)\right)^{\frac{1}{2}}, \qquad p(0) = 1,$$

is well defined and analytic.

We define the **modified Fourier transform** by

$$(4') \qquad \tilde{\phi}(F) = \int_{\mathfrak{g}} e^{2\pi i \langle F, X\rangle} \phi(\exp X) p(X) dX.$$

For unimodular exponential groups the Fourier transform is a unitary bijection of $L^2(G, dg)$ to $L^2(\mathfrak{g}^*, dF)$ and we shall use it instead of (4) in expression (5).

Also, we can replace the differential $2r$-form $\frac{\sigma^r}{r!}$ by the non-homogeneous form $e^\sigma = 1 + \sigma + \frac{\sigma^2}{2} + \dots$ and agree that the integral of the non-homogeneous form over a k-dimensional manifold depends only on the homogeneous component of degree k.

Then the **modified Rule 6** acquires the elegant form

$$(7) \qquad \operatorname{tr} T_\Omega(\exp X) = \frac{1}{p(X)} \int_\Omega e^{2\pi i \langle F, X\rangle + \sigma}.$$

Since $p(X) \equiv 1$ for nilpotent groups, the modified Rule 6 coincides in this case with the original one.

It turns out that the modified Rule 6 is valid for a wide class of non-nilpotent groups provided that test functions are subjected to additional restrictions.

Remark 1. Consider the function j of one complex variable z defined by

$$(8) \qquad j^2(z) = \frac{\sinh \frac{z}{2}}{\frac{z}{2}}, \qquad j(0) = 1.$$

It is a holomorphic function in the disc $|z| < 2\pi$ with the Taylor decomposition

$$j(z) = 1 + \frac{z^2}{48} + \frac{z^4}{23040} + \dots,$$

and it has a two-valued analytic continuation to the whole complex plane except at the branching points $z_n = 2\pi i n,\ n \in \mathbb{Z}\backslash\{0\}$.

The function $p(X)$ that we used in the modified Fourier transform can be expressed in terms of j as follows. Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of the operator $\operatorname{ad} X$. Then

$$(9) \quad p(X) = \prod_{k=1}^{n} j(\lambda_k) = 1 + \frac{\operatorname{tr}(\operatorname{ad} X)^2}{48} + \frac{(\operatorname{tr}(\operatorname{ad} X)^2)^2}{4608} - \frac{7 \operatorname{tr}(\operatorname{ad} X)^4}{5760} + \dots.$$

Maxim Kontsevich observed that the well-known identity

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

implies the equality

$$j^{-2}(z) = \Gamma\left(1 + \frac{z}{2\pi i}\right)\Gamma\left(1 - \frac{z}{2\pi i}\right).$$

Since $\operatorname{ad} X$ is a real operator, its eigenvalues are either real or split into complex conjugate pairs. Taking into account the property $\Gamma(\overline{z}) = \overline{\Gamma(z)}$, we conclude that

$$\frac{1}{p(X)} = \prod_{k=1}^{n} j^{-1}(\lambda_k) = \left| \prod_{k=1}^{n} \Gamma\left(1 + \frac{\lambda_k}{2\pi i}\right) \right|.$$

This strongly suggests that we introduce the function

$$k(X) = \prod_{k=1}^{n} \frac{1}{\Gamma\left(1 + \frac{\lambda_k}{2\pi i}\right)} = \frac{1}{\det \Pi(\frac{\operatorname{ad} X}{2\pi i})}$$

where $\Pi(z) = \Gamma(1+z)$, and use it instead of $p(X)$ in the definition of the modified Fourier transform.

The advantage of this replacement is that $\frac{1}{\Gamma(z)}$ is an entire function, hence $k(X)$ is well defined on $\mathfrak{g}$ and even on its complexification. ♡

Example 2. Let $G = \mathrm{Aff}_+(1,\mathbb{R})$ be the group of orientation preserving affine transformations of the real line.[2] The group G is a semidirect product of the subgroup A of dilations and the normal subgroup B of translations. It has a matrix realization by 2×2 matrices of the form

$$g_{a,b} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad a, b \in \mathbb{R},\ a > 0.$$

The elements of the Lie algebra $\mathfrak{g}$ and of the dual space $\mathfrak{g}^*$ are realized by real 2×2 matrices of the form

$$X = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}$$

and the coadjoint action is

$$K(a,b)\colon\ (x,y) \mapsto (x + a^{-1}by,\ a^{-1}y).$$

We see that $\mathfrak{g}^*$ splits into two 2-dimensional open orbits $\Omega_\pm = \{(x,y) \mid \pm y > 0\}$ and a family of 0-dimensional orbits $\Omega_x = \{(x,0)\}$.

Exercise 1. Compute the exponential map for the group G.

Hint. For $\alpha \neq 0$ use the identity $\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} = Q \cdot \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \cdot Q^{-1}$ for an appropriate invertible matrix Q.

Answer: $\quad \exp\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^\alpha & \frac{e^\alpha - 1}{\alpha}\beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^\alpha & e^{\alpha/2} j^2(\alpha)\beta \\ 0 & 1 \end{pmatrix}.$

Now we compute the modified Fourier transform (4′) for the group G. We start with a function $\phi \in \mathcal{A}(G)$. Its modified Fourier transform is a function $\psi = \widetilde{\phi}$ on $\mathfrak{g}^*$ given by

$$\psi(x,\,y) = \int_{\mathbb{R}^2} e^{2\pi i(\alpha x + \beta y)} \psi\left(e^\alpha,\ \frac{e^\alpha - 1}{\alpha}\beta\right) \cdot \sqrt{\frac{\alpha/2}{\sinh(\alpha/2)}}\, d\alpha d\beta.$$

The unirreps of G corresponding to the open orbits $\Omega_\pm$ are given by a simple formula. We omit the standard computations according to Rule 2 and write down the final result.

[2] We refer to [**LL**] where this simple but instructive case is treated in detail (see also [**BCD**] and [**Ki2**]).

Namely, consider the space $L^2(\mathbb{R}^\times, \frac{dx}{x})$, the natural Hilbert space associated with the smooth manifold $R^\times = \mathbb{R}\backslash\{0\}$ (see Appendix V.2.2). The unitary action of G on this space has the form

$$\big(\pi(g_{a,b})f\big)(x) = e^{2\pi ibx}f(ax).$$

This representation is manifestly reducible: the functions concentrated on positive and negative parts of $\mathbb{R}^\times$ form two invariant subspaces. These subspaces are irreducible and the corresponding unitary representations are exactly $\pi_\pm$.

Let us compute the generalized characters of these representations. For a test measure $\phi(a, b)\frac{dadb}{a}$ with $\phi \in \mathcal{A}(G)$ we have

$$\begin{aligned}\big(\pi(\phi)f\big)(x) &= \int_{\substack{a\in\mathbb{R}^\times\\ b\in\mathbb{R}}} \phi(a, b)e^{2\pi ibx}f(ax)\frac{dadb}{a}\\ &= \int_{\substack{y\in\mathbb{R}^\times\\ b\in\mathbb{R}}} \phi\left(\frac{y}{x}, b\right)e^{2\pi ibx}f(y)\frac{dydb}{y}.\end{aligned}$$

So, the kernel of the operator $\pi_\pm(\phi)$ is

$$K_\phi^\pm(x, y) = \int_{\mathbb{R}} \phi\left(\frac{y}{x}, b\right)e^{2\pi ibx}db, \quad \pm y > 0, \ \pm x > 0.$$

For the trace of $\pi_\pm(\phi)$ we get the expression

$$\operatorname{tr}\pi_\pm(\phi) = \int_{\mathbb{R}_\pm^\times} K_\phi^\pm(x, x)\frac{dx}{x} = \int_{\mathbb{R}\times\mathbb{R}_\pm^\times} \phi(1, b)e^{2\pi ibx}db\frac{dx}{x}.$$

In terms of the Fourier transform $\widetilde{\phi} = \psi$ the last expression can be written as

$$\operatorname{tr}\pi_\pm(\widetilde{\psi}) = \int_{\mathbb{R}_\pm^2} \psi(x, y)\frac{dxdy}{y} = \int_{\Omega_\pm} \psi(F)\sigma(F)$$

in perfect accordance with Rule 6. $\diamondsuit$

1.5. Infinitesimal characters.

The initial formulation of Rule 7 of the User's Guide also must be corrected. Recall that in Section 2.5 of Chapter 3 we defined the bijection $A \longleftrightarrow P_A$ between $Z(\mathfrak{g})$ and $Pol(\mathfrak{g}^*)^G$ using the symmetrization map **sym**. For a homogeneous element $A \in Z(\mathfrak{g})$ we have

$$A = (2\pi i)^{\deg P_A}\mathbf{sym}(P_A).$$

The point is that this map is the isomorphism of linear spaces, but in general *not an algebra homomorphism.*

It forces us to modify Rule 7 since both the maps $P \mapsto P(\Omega)$ and $A \mapsto I_\pi(A)$ are algebra homomorphisms. We shall have an opportunity to speak more about this in Section 3.6 in Chapter 5 and Section 6.2 in Chapter 6 and here only give the correct formula.

Namely, the **modified Rule 7** looks exactly as before:

$$I_{\pi_\Omega}(A) = P_A(\Omega),$$

but with a differently defined correspondence $A \longleftrightarrow P_A^{new}$.

To define this new correspondence we consider the Taylor decomposition of the function $p(X)$ defined by (6) and (9). The formal Fourier transform associates to this power series a formal differential operator J of infinite order with constant coefficients on $\mathfrak{g}^*$. This operator acts on $Pol(\mathfrak{g}^*)$ and is invertible.

If $\{X_i\}$ and $\{F^j\}$ are any dual bases in $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively, then J is obtained from $p(X)$ by substituting $\frac{\partial}{\partial F^i}$ instead of X_i.

For $A \in U(\mathfrak{g})$ we now define $P_A^{new} \in Pol(\mathfrak{g}^*)$ so that

$$(10) \qquad A = (2\pi i)^{\deg P_A}\mathbf{sym}(JP_A^{new}) \qquad \text{or} \qquad P_A^{new} = J^{-1}P_A^{old}.$$

Note that for nilpotent $\mathfrak{g}$ the definition of P_A^{new} coincides with that of P_A^{old} because $p(X) \equiv 1$ and $J = 1$ in this case.

2. General solvable Lie groups

2.1. Tame and wild Lie groups.

There is a new phenomenon that occurs when we consider non-exponential solvable Lie groups. Some of these groups are wild, i.e. do not belong to type I in the sense of von Neumann.

This means that the representation theory for these groups has several unpleasant features. See Appendix IV.2.6 for details.

It is interesting that there are two different reasons why a solvable Lie group may be wild. Both have a simple interpretation in the orbit picture.

The first reason is that the space $\mathcal{O}(G)$ can violate the separation axiom T_0. This is because coadjoint orbits are not necessarily closed, even locally. The simplest example of such an occurrence was discovered by F.I. Mautner in the 1950's and was rediscovered later many times. We describe it here.

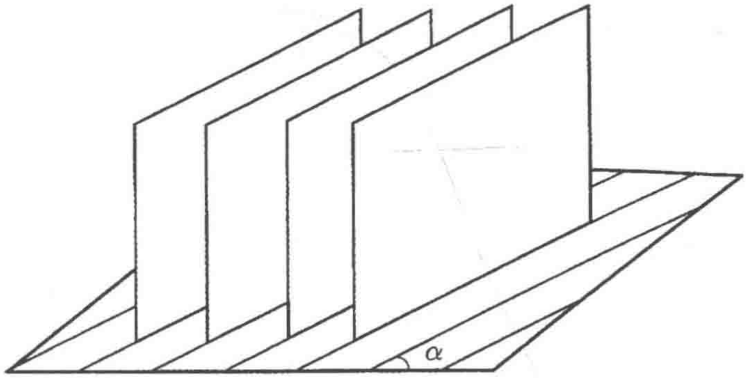

Figure 1. Coadjoint orbits for G_α.

Example 3. Consider the family of 5-dimensional Lie groups G_α that depend on a real parameter α and have the following matrix realization:

$$G_\alpha \ni g(t,\, a,\, b) = \begin{pmatrix} e^{it} & 0 & a \\ 0 & e^{i\alpha t} & b \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where } t \in \mathbb{R},\ a, b \in \mathbb{C}. \tag{11}$$

It turns out that G_α is tame if and only if α is a rational number.

The Lie algebra $\mathfrak{g}_\alpha$ consists of matrices $\begin{pmatrix} it & 0 & a \\ 0 & i\alpha t & b \\ 0 & 0 & 0 \end{pmatrix}$, and the elements of the dual space $\mathfrak{g}^*$ can be realized by matrices

$$F_{z,w,\tau} = \begin{pmatrix} i\tau & 0 & 0 \\ 0 & 0 & \\ z & w & 0 \end{pmatrix}.$$

The coadjoint action in the coordinates z, w, τ has the form

$$K\big(g(t,\, a,\, b)\big)\big(z,\, w,\, \tau\big) = \big(e^{-it}z,\, e^{-i\alpha t}w,\, \tau + \operatorname{Im}(az + \alpha bw)\big).$$

A picture of the coadjoint orbits for irrational α is shown in Figure 1.

We see that the topological space $\mathcal{O}(G_\alpha)$ does not satisfy the axiom T_0.

Exercise 2.* Consider two families of representations of G_α: the family $U_{a,b}$, $a, b \in \mathbb{C}$, acting in $L^2(\mathbb{R},\, d\tau)$ by the formula

$$(U_{a,b}(t,\, z,\, w)\phi)(\tau) = e^{2\pi i \operatorname{Re}(e^{it}az + e^{i\alpha t}bw)}\phi(\tau + t),$$

and the family $V_{r_1,r_2,s}$, $r_1, r_2 \in \mathbb{R}_+$, $s \in \mathbb{R}$, acting in $L^2(\mathbb{T}^2, d\theta_1 d\theta_2)$ by the formula

$$\begin{aligned}(V_{r_1,r_2,s}(t,\, z,\, w)\psi)(\theta_1,\, \theta_2) \\ = e^{2\pi i \operatorname{Re}(ts + e^{it}r_1 z + e^{i\alpha t}r_2 w)}\psi(\theta_1 + t \bmod 1,\, \theta_2 + \alpha t \bmod 1).\end{aligned}$$

a) Show that there are two different decompositions of the right regular representation ρ of G into irreducible components:

$$\rho = \int_{\mathbb{C}\times\mathbb{C}} U_{a,b}\, da\, d\bar{a}\, db\, d\bar{b} \tag{12}$$

and

$$\rho = \int_{\mathbb{R}^2_+\times\mathbb{R}} V_{r_1,r_2,s}\, dr_1\, dr_2\, ds. \tag{13}$$

b) Show that no $U_{a,b}$ is equivalent to any $V_{r_1,r_2,s}$.

Hint. See [**Ki2**], Part III, §19. ♣ ◇

The second reason is that the canonical form σ on some orbits can be non-exact. This innocent looking circumstance implies that after applying the Mackey Inducibility Criterion (see Appendix V.2.4 or [**Ki2**], §13) we run into representations of some non-abelian discrete groups that are usually wild.

Example 4. The simplest group for which this situation occurs has dimension seven and can be realized by block-diagonal 6×6 matrices with diagonal 3×3 blocks of the form

$$\begin{pmatrix} e^{is} & 0 & z \\ 0 & e^{it} & w \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & s & r \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad s, r, t \in \mathbb{R},\ z, w \in \mathbb{C}. \tag{14}$$

We give here the description of the orbits and representations of this group following [**Ki2**], §19.

Let us introduce the seven real coordinates x, y, u, v, s, t, r on $\mathfrak{g}$ so that $z = x+iy$, $w = u+iv$. We denote by X, Y, U, V, S, T, R the corresponding basic vectors in $\mathfrak{g}$ which also serve as dual coordinates in $\mathfrak{g}^*$. The non-zero commutators are

$$[S, X] = Y, \quad [S, Y] = -X, \quad [T, U] = V, \quad [T, V] = -U, \quad [S, T] = R.$$

Let G be a connected and simply connected Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$.

Exercise 3. Show that the coadjoint orbits of maximal dimension in $\mathfrak{g}^*$ are diffeomorphic to $\mathbb{T}^2\times\mathbb{R}^2$ and the canonical symplectic form σ is not exact.

Hint. Check that for non-zero r_1, r_2, r_3 the equations

$$X^2+Y^2 = r_1^2, \qquad U^2+V^2 = r_2^2, \qquad R = r_3$$

define a coadjoint orbit with canonical 2-form

$$\sigma = d\phi \wedge dS + d\psi \wedge dT + r_3\, d\phi \wedge d\psi$$

where the angle coordinates ϕ and ψ are defined by

$$X = r_1 \cos\phi, \quad Y = r_1 \sin\phi, \quad U = r_2 \cos\psi, \quad V = r_2 \sin\psi.$$

♣

The fact that G is wild can be established in the following way. There is an abelian normal subgroup $A \subset G$ whose Lie algebra $\mathfrak{a}$ is spanned by X, Y, U, V. The Pontrjagin dual group $\widehat{A}$ consists of characters

$$\chi(\exp(xX + yY + uU + vV)) = e^{2\pi i(ax+by+cu+dv)}.$$

The splitting of $\widehat{A}$ into G-orbits is tame: orbits are given by the equations

$$a^2 + b^2 = r_1^2, \qquad c^2 + d^2 = r_2^2.$$

So we can apply the Inducibility Criterion in Appendix V.2.4 to this situation. The conclusion is that a unirrep π of G has the form

$$\pi_{\chi,\rho} = \operatorname{Ind}_{H_\chi}^G \rho$$

where $\chi \in \widehat{A}$, H_χ is the stabilizer of the point χ in G, and ρ is a unirrep of H_χ such that $\rho(a) = \chi(a) \cdot 1$ for $a \in A \subset H_\chi$.

The representation in question remains in the same equivalence class when we replace the pair (χ, ρ) by $g \cdot (\chi, \rho) = (\chi \circ A(g^{-1}), \rho \circ A(g^{-1}))$ where $A(g) : x \mapsto gxg^{-1}$. So, unirreps of G are actually labelled by the following data: a G-orbit $\Omega \subset \widehat{A}$ and a unirrep ρ of the stabilizer H_χ of some point $\chi \in \Omega$ (subjected to the condition $\rho|_A = \chi \cdot 1$).

Call χ non-degenerate if $a^2 + b^2 \neq 0$ and $c^2 + d^2 \neq 0$. Let $\widehat{A}_0$ be the set of all non-degenerate characters. The stabilizer H_χ of $\chi \in \widehat{A}_0$ in G actually does not depend on χ. It is the semidirect product

$$H_\chi = H_{\mathbb{Z}} \ltimes A \quad \text{where } H_{\mathbb{Z}} := \exp(\mathbb{Z} \cdot S + \mathbb{Z} \cdot T + \mathbb{R} \cdot R).$$

Let $\widehat{G}_0$ be the open part of $\widehat{G}$ that consists of representations whose restrictions to A split into non-degenerate characters. It follows from above that $\widehat{G}_0$ is in fact homeomorphic to $(\widehat{A}_0)_G \times \widehat{H_{\mathbb{Z}}}$. We see that G and $H_{\mathbb{Z}}$ are of the same type: either both are tame or both are wild. So, we have to show that $H_{\mathbb{Z}}$ is a wild group.

The group $H_{\mathbb{Z}}$ is a subgroup of the Heisenberg group that consists of matrices

$$h(m,n,r) = \begin{pmatrix} 1 & m & r \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{with } m,n \in \mathbb{Z},\ r \in \mathbb{R}.$$

It has a normal abelian subgroup B defined by the condition $n = 0$. A generic character χ of B has the form

$$\chi_{\theta,c}(h(m,0,r)) = e^{2\pi i(m\theta + rc)}, \quad \theta \in \mathbb{R}/\mathbb{Z},\ c \in \mathbb{R}.$$

The group H acts on $\widehat{B}$ and this action factors through H/B since B acts trivially on $\widehat{B}$. An easy computation shows that the generator of $H/B \cong \mathbb{Z}$ defines the transformation

$$(\theta,\ c) \mapsto (\theta + c \bmod 1,\ c).$$

We see that the partition of $\widehat{B}$ into H-orbits is wild: for any irrational c the set $\theta + \mathbb{Z} \cdot c$ mod 1 is dense in $\mathbb{R}/\mathbb{Z}$.

Consider the representations $\pi_{\theta,c} = \mathrm{Ind}_B^H \chi_{\theta,c}$ of H.

Exercise 4. a) Show that $\pi_{\theta,c}$ and $\pi_{\theta',c'}$ are equivalent iff $c' = c$ and $\theta' = \theta + kc \bmod 1$ for some $k \in \mathbb{Z}$.

b) Show that for c irrational all the points $\pi_{\theta',c} \in \widehat{H}$ belong to any neighborhood of $\pi_{\theta,c}$.

Hint. Write the explicit formula for representations and try to find the intertwining operators between them. ♣ ♢

Remark 2. It is natural for wild groups of the first kind to extend the notion of coadjoint orbit and consider ergodic G-invariant measures on $\mathfrak{g}^*$ as **virtual coadjoint orbits.**[3]

The notion of a virtual subgroup was suggested by Mackey. The idea to use it in the orbit method was proposed in [**K2**]. Soon after that I learned that this idea had already been realized by Pukanszky.

In particular, the analogue of the integral formula for generalized characters was obtained in [**P2**]. The left-hand side of this formula is the **relative trace** of the operator $\pi(g)$ in the sense of von Neumann and the right-hand side is the integral over a virtual orbit. ♡

[3] One can prove that when $\mathcal{O}(G)$ satisfies the separation axiom T_0 all such measures are proportional to canonical measures on orbits.

2.2. Tame solvable Lie groups.

Now we come back to the tame solvable Lie groups. The remarkable fact is that the User's Guide still works in this case after appropriate amendments to all rules except 7 and 9.

Almost all the results described in this section are due to Louis Auslander and Bertram Kostant [**AK**]. We shall present these results here together with some simplifications and complements suggested by I.M. Shchepochkina [**Sh**].[4]

We start with the following simple criterion.

Theorem 2 (Auslander-Kostant). *A connected and simply connected solvable Lie group G is tame (that is, of type* I*) iff the following two conditions are fulfilled:*

1. *The space $\mathcal{O}(G)$ satisfies the separation axiom T_0.*
2. *The canonical form σ is exact on each orbit.* □

From now on we assume that G satisfies these conditions.

Even for tame solvable groups the correspondence between coadjoint orbits and representations need not be one-to-one. To describe this correspondence we have to modify the space $\mathfrak{g}^*$.

The point is that, unlike the case of exponential Lie groups, a coadjoint orbit Ω can be topologically non-trivial; in particular, the Betti numbers $b_1(\Omega)$ and $b_2(\Omega)$ can be non-zero.

A simple topological consideration (see Appendices I.2.3 and III.4.2) shows that the fundamental group $\pi_1(\Omega)$ is isomorphic to $Stab(F)/Stab^0(F)$.

We define a **rigged momentum** for a Lie group G as a pair (F, χ) where $F \in \mathfrak{g}^*$ and χ is a unitary 1-dimensional representation of $Stab(F)$ such that

$$\chi_*(e) = 2\pi i F\,|_{Stab(F)}\,. \tag{15}$$

Recall that such a χ exists only if the orbit Ω_F is integral (see Proposition 2 in Section 1.2.4). For the type I solvable Lie groups this condition is always satisfied because σ is exact, hence all orbits are integral.

The set of all rigged momenta will be denoted by $\mathfrak{g}^*_{rigg}$. The group G acts naturally on $\mathfrak{g}^*_{rigg}$ and this action commutes with the projection $\Pi\colon\ \mathfrak{g}^*_{rigg} \to \mathfrak{g}^*$ given by $\Pi(F, \chi) = F$.

The G-orbits in $\mathfrak{g}^*_{rigg}$ will be called **rigged coadjoint orbits** and the set of all such orbits will be denoted by $\mathcal{O}_{rigg}(G)$.

[4]Unfortunately, the complete text of her Ph.D. thesis (Moscow, 1980) was never published and so is still inaccessible to the mathematical community.

It is worthwhile to mention that for tame solvable Lie groups the projection Π is onto and the fiber over a point $F \in \mathfrak{g}^*$ is a torus of dimension equal to the first Betti number of Ω_F. So, the correspondence between usual and rigged orbits is one-to-many.

The **modified Rule 1** is given by

Theorem 3 (Auslander–Kostant). *For any connected and simply connected solvable Lie group G of type* I *there is a natural bijection between the set $\widehat{G}$ of unirreps and the space $\mathcal{O}_{rigg}(G)$ of rigged coadjoint orbits.* □

We refer to the original paper [**AK**] for the detailed proof and describe here only the construction of a unirrep T_Ω associated with a rigged orbit Ω.

It turns out that it requires a new procedure of **holomorphic induction** and we outline the main idea behind this notion. We shall speak again about it in Chapter 5.

From now on we assume that $\Omega \in \mathcal{O}_{rigg}$ and $(F, \chi) \in \Omega$.

We start with the case when a real polarization $\mathfrak{h}$ for F exists and satisfies the Pukanszky condition. Then the usual induction procedure of Rule 2 of the User's Guide is applicable with a minor modification.

Namely, let $H^0 = \exp \mathfrak{h}$, and let H be the group generated by H^0 and $Stab(F)$. (Note that $Stab^0(F)$ is contained in H^0.) We define the 1-dimensional unirrep $U_{F,\chi,H}$ of H by

$$U_{F,\chi,H}(g) = \begin{cases} e^{2\pi i \langle F, X \rangle} & \text{for } g = \exp X, \ X \in \mathfrak{h}, \\ \chi(g) & \text{for } g \in Stab(F). \end{cases} \tag{16}$$

The correctness of this definition is ensured by (15).

Then we define the desired unirrep π_Ω by the formula

$$\pi_\Omega = \operatorname{Ind}_H^G U_{F,\chi,H}, \tag{17}$$

which can be considered as the **first amendment to Rule 2**.

This procedure is not sufficient for the construction of unirreps for all rigged orbits. The reason is that for some $F \in \mathfrak{g}^*$ there is no real polarization, i.e. no subalgebra $\mathfrak{h} \subset \mathfrak{g}$ that is subordinate to the functional F and has the required codimension $\frac{1}{2}\mathrm{rk} B_F$.

On the other hand, a complex polarization, i.e. a complex subalgebra $\mathfrak{p} \subset \mathfrak{g}_\mathbb{C}$ with these properties, always exists. This follows from the procedure described in Section 5.2 of Chapter 1. We keep the notation of this section.

The definition of the induction procedure can be further modified so that a complex polarization $\mathfrak{p}$ is used instead of a real one. Namely, let $\mathfrak{p}$ be a complex polarization for $F \in \mathfrak{g}^*$.

Following the procedure described in Section 5.1 of Chapter 1, we introduce the real subspaces $\mathfrak{e} \subset \mathfrak{g}$ and $\mathfrak{d} \subset \mathfrak{g}$ such that their complexifications are $\mathfrak{p} + \overline{\mathfrak{p}}$ and $\mathfrak{p} \cap \overline{\mathfrak{p}}$, respectively.

Note that $\mathfrak{d}$ is always a subalgebra in $\mathfrak{g}$ that contains $stab(F) = \operatorname{Lie}(Stab(F))$. Let D be the corresponding connected subgroup in G and put $H = D \cdot Stab(F)$. It is clear that $H^0 = D$.

We call the polarization $\mathfrak{p}$ **admissible** if $\mathfrak{e} \subset \mathfrak{g}$ is also a subalgebra and denote by E the corresponding subgroup in G.

Recall that for any $X \in \mathfrak{g}$ there corresponds a right-invariant vector field $\widehat{X}$ on G. We extend the map $X \mapsto \widehat{X}$ to a complex-linear map from $\mathfrak{g}_{\mathbb{C}}$ to $Vect_{\mathbb{C}}(G)$ and use the same notation for the extended map.

Consider the space $L(G, F, \mathfrak{p})$ of smooth functions on G satisfying the conditions

$$
\begin{gathered}
(\widehat{X} - 2\pi i \langle F, X \rangle) f = 0 \quad \text{for all } X \in \mathfrak{p}, \\
f(hg) = \chi(h) f(g) \text{ for all } h \in H.
\end{gathered}
\tag{18}
$$

Note that for $X \in \mathfrak{g}$ the real field $\widehat{X}$ is the infinitesimal left shift along the subgroup $\exp(\mathbb{R}X)$. So, the first condition in (18) is just the infinitesimal form of the equation $f(\exp X \cdot g) = e^{2\pi i \langle F, X \rangle} f(g)$. We observe also that the second condition in (18) for $h \in H^0$ follows from the first one due to (15).

From the geometric point of view elements of $L(G, F, \mathfrak{p})$ can be considered as smooth sections of a certain line bundle L over the right homogeneous space $M = H \backslash G$ with an additional condition: they are holomorphic along some complex submanifolds isomorphic to $H \backslash E$ (cf. Section 5.1 of Chapter 1). Moreover, L is a G-bundle and the action of $Stab(F)$ on the 1-dimensional fiber of L over F is given by the character χ.

We denote by $L_0(G, F, \mathfrak{p})$ the subspace of those sections whose supports have compact projections on $E \backslash G$. One can introduce in $L_0(G, F, \mathfrak{p})$ a G-invariant inner product (see the details in the worked-out example below). The completion of $L_0(G, F, \mathfrak{p})$ with respect to this inner product is a Hilbert space $\mathcal{H}$ where G acts by unitary operators.

It turns out that under a suitable positivity condition on $\mathfrak{p}$ the resulting representation is irreducible and depends only on the rigged orbit Ω that contains (F, χ). We denote it by π_Ω and consider the described construction as the **second amendment to Rule 2**.

The most important particular case of this construction is when $\mathfrak{p}$ is a so-called **Kähler polarization**. This means that

1. $\mathfrak{p} + \overline{\mathfrak{p}} = \mathfrak{g}_{\mathbb{C}}$.
2. $\mathfrak{p} \cap \overline{\mathfrak{p}} = \mathfrak{h}_{\mathbb{C}}$ where $\mathfrak{h} = stab(F)$.

3. The Hermitian form $h(X, Y) := \frac{1}{2i} B_F(X, \overline{Y})$ is positive definite on $\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}$.

In this case $E = G$ and $\mathfrak{p}$ defines a G-invariant complex structure on the manifold $M = H\backslash G$. Moreover, M as a homogeneous manifold is isomorphic to a coadjoint orbit $\Omega \subset \mathfrak{g}^*$, hence possesses a canonical volume form. One can introduce a G-invariant Hermitian structure on L so that the representation π_Ω acts on the space of square integrable holomorphic sections of L.

The practical application of this procedure is explained in detail in the next section with a typical example.

3. Example: The diamond Lie algebra $\mathfrak{g}$

We illustrate the general theory described above using a concrete example.

3.1. The coadjoint orbits for $\mathfrak{g}$.

Let $\mathfrak{g}$ be the so-called **diamond Lie algebra $\mathfrak{g}$** with basis T, X, Y, Z and non-zero commutation relations

$$(19) \qquad [T, X] = Y, \qquad [T, Y] = -X, \qquad [X, Y] = Z.$$

It admits the matrix realization as a Lie subalgebra of $\mathfrak{sp}(4, \mathbb{R})$ with a generic element

$$(20) \qquad S = \theta T + aX + bY + cZ = \begin{pmatrix} 0 & a & b & 2c \\ 0 & 0 & -\theta & b \\ 0 & \theta & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To describe the corresponding simply connected Lie group G, we consider the group $E(2)$ of orientation preserving rigid motions of the Euclidean plane $\mathbb{R}^2$. As a smooth manifold, $E(2)$ is isomorphic to $S^1 \times \mathbb{R}^2$. It turns out that our group G can be viewed as a universal covering of a central extension of $E(2)$ (see below).

Let us transfer the coordinates (θ, a, b, c) in $\mathfrak{g}$ to the group G by

$$(21) \qquad g(\theta, a, b, c) = \exp(\theta T + cZ)\exp(aX + bY).$$

Warning. The group G admits a global coordinate system and is diffeomorphic to $\mathbb{R}^4$ but it is *not exponential.* ♠

Exercise 5. a) Show that the center C of G consists of elements $g(\theta, a, b, c)$ satisfying the conditions: $a = b = 0$, $\theta \in 2\pi\mathbb{Z}$, $c \in \mathbb{R}$.

b) Check that the adjoint group G/C is isomorphic to the motion group $E(2)$.

Hint. Use the explicit formula for the coadjoint action given below. ♣

The dual space $\mathfrak{g}^*$ can be conveniently realized as the set of matrices

$$F(t,\, x,\, y,\, z) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & t & 0 \\ y & -t & 0 & 0 \\ z & y & -x & 0 \end{pmatrix} \tag{22}$$

so that $\langle F,\, S\rangle = \operatorname{tr}(FS) = \theta t + ax + by + cz$.

The explicit form of the coadjoint action is:

$$K(g(0,a,b,c))\, F(t,x,y,z) = F\left(t - bx + ay - \frac{a^2+b^2}{2}z, x + bz, y - az, z\right);$$
$$K(g(\theta,0,0,0))\, F(t,x,y,z) = F(t, x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z).$$

The invariant polynomials on $\mathfrak{g}^*$ are z and $x^2 + y^2 + 2tz$. Hence, the generic orbits are 2-dimensional paraboloids

$$\Omega_{c_1,c_2}: \qquad z = c_1 \neq 0, \quad t = c_2 - \frac{x^2+y^2}{2c_1}. \tag{23a}$$

The hyperplane $z = 0$ splits into 2-dimensional cylinders

$$\Omega_r: \qquad x^2 + y^2 = r^2, \quad r > 0, \tag{23b}$$

and 0-dimensional orbits (fixed points)

$$\Omega_0^c = (0,\, 0,\, 0,\, c), \qquad c \in \mathbb{R}. \tag{23c}$$

The quotient topology in $\mathcal{O}(G)$ is the ordinary topology on each piece (23a), (23b), (23c):

$$\{\Omega_{c_1,c_2}\} \cong \mathbb{R}^\times \times \mathbb{R}, \qquad \{\Omega_r\} \cong \mathbb{R}_{>0}, \qquad \{\Omega_0^c\} \cong \mathbb{R}.$$

As $c_1 \to 0$ and $2c_1c_2 \to r^2$, the paraboloid Ω_{c_1,c_2} tends to the cylinder Ω_r; when $c_1 \to 0$ and $c_1c_2 \to 0$, the limit is the set of all Ω_0^c.

So, to obtain the topological space $\mathcal{O}(G)$ one should take the real plane $\mathbb{R}^2$ with coordinates $(a = c_1,\, b = 2c_1c_2)$, delete the ray $a = 0,\, b \leq 0$, and paste in a line $\mathbb{R}$ instead of the deleted origin. To get the space $\mathcal{O}_{rigg}(G)$, we have in addition to replace each point of the open ray $a = 0,\, b > 0$ by a circle.

3.2. Representations corresponding to generic orbits.

All generic coadjoint orbits are simply connected. Hence, there is no difference between orbits and rigged orbits and, according to Rule 1 of the User's Guide, to any generic orbit Ω_{c_1,c_2} there corresponds exactly one (up to equivalence) unirrep π_{c_1,c_2}. We describe here how to construct these unirreps.

It is not difficult to show that for $c_1 \neq 0$ the functional F has no real polarization. Indeed, the only 3-dimensional subalgebra of $\mathfrak{g}$ is spanned by X, Y, Z, hence is not subordinate to F.

However, there exist two 3-dimensional complex subalgebras $\mathfrak{p}_\pm \subset \mathfrak{g}_\mathbb{C}$ that are subordinate to F. To simplify the computations, let us choose the special point $F \in \Omega_{c_1,c_2}$ with coordinates $(0,\, 0,\, c_1,\, c_2)$. Then we can put

$$\mathfrak{p}_\pm = \mathbb{C}\cdot T \oplus \mathbb{C}\cdot (X \pm iY) \oplus \mathbb{C}\cdot Z.$$

Indeed, since $[\mathfrak{p}_\pm,\, \mathfrak{p}_\pm] = \mathbb{C}\cdot(X \pm iY)$, we see that $F\big|_{[\mathfrak{p}_\pm,\, \mathfrak{p}_\pm]} = 0$. Hence, both $\mathfrak{p}_\pm$ are complex polarizations of F. Later on we concentrate on $\mathfrak{p}_+$, which will be denoted simply by $\mathfrak{p}$, and only mention the necessary modifications for $\mathfrak{p}_-$.

In this case we have $\mathfrak{p} + \overline{\mathfrak{p}} = \mathfrak{g}$, $\mathfrak{p} \cap \overline{\mathfrak{p}} = \mathbb{C}\cdot T \oplus \mathbb{C}\cdot Z$. The 1-dimensional complex space $\mathfrak{g}_\mathbb{C}/\mathfrak{p}$ is spanned by the vector $\xi = (X - iY) \bmod \mathfrak{p}$ and the Hermitian form h is given by $h(\xi,\, \xi) = c_1$. So, $\mathfrak{p}$ is Kähler iff $c_1 > 0$.

Let $D = \exp(\mathbb{R}\cdot T \oplus \mathbb{R}\cdot Z)$. The manifold $M = D\backslash G$ is an ordinary plane $\mathbb{R}^2$ with coordinates $(u,\, v)$. The right action of G on M factors through G/C and produces exactly the transformation group $E(2)$:

$$\begin{aligned}(u,\, v)\cdot \exp(\theta T) &= (u\cos\theta + v\sin\theta,\, -u\sin\theta + v\cos\theta) \quad \text{(rotations)};\\ (u,\, v)\cdot \exp(aX + bY + cZ) &= (u + a,\, v + b) \quad \text{(translations)}.\end{aligned}$$

Exercise 6. a) Establish a G-equivariant isomorphism α between the manifold M and the coadjoint orbit Ω_{c_1,c_2} with $c_1 \neq 0$.

b) Compute the image $\alpha^*(\sigma)$ of the canonical symplectic form σ under this isomorphism.

Hint. a) Take into account that Ω_{c_1,c_2} is a left G-manifold while M is a right G-manifold. So, the equivariance of the map $\alpha : M \to \Omega_{c_1,c_2}$ is expressed by the equality $g\cdot\alpha(u,\, v) = \alpha((u,\, v)\cdot g^{-1})$.

b) Use the fact that, on linear functions on $\mathfrak{g}^*$, Poisson brackets define the structure of a Lie algebra isomorphic to $\mathfrak{g}$.

Answer: a) $\alpha(u,\, v) = F(-c_1 v,\, c_1 u,\, c_1,\, c_2 - c_1\frac{u^2+v^2}{2})$.

b) $\alpha^*(\sigma) = c_1\, du \wedge dv$.

According to the general theory, every complex polarization $\mathfrak{p}$ defines a G-invariant complex structure on M. In our case this structure is given by the global complex coordinate $w = u + iv$. ♣

Proposition 3. *The space $L(G, F, \mathfrak{p})$ given by the system (18) coincides in our case with the space of holomorphic sections of a (topologically trivial) holomorphic line bundle over M.*

Proof. In coordinates (21) the right invariant vector fields on G are given by

$$\widehat{X} = \cos\theta \cdot \partial_a - \sin\theta \cdot \partial_b + \frac{b\cos\theta + a\sin\theta}{2} \cdot \partial_c; \qquad \widehat{Z} = \partial_c;$$
$$\widehat{Y} = \sin\theta \cdot \partial_a + \cos\theta \cdot \partial_b + \frac{b\sin\theta - a\cos\theta}{2} \cdot \partial_c; \qquad \widehat{T} = \partial_\theta.$$

Thus, the second condition in (18) (the "real part" of it) takes the form

$$\frac{\partial f}{\partial c} = 2\pi i c_1 f, \qquad \frac{\partial f}{\partial \theta} = 2\pi i c_2 f.$$

This condition is equivalent to the equality

$$f(\theta, a, b, c) = e^{2\pi i (c_1 c + c_2 \theta)} f(0, a, b, 0). \tag{24}$$

Let us define the section $s\colon M \to G\colon (u, v) \mapsto g(0, u, v, 0)$.

The equality above implies that a solution f as a function on G is completely determined by its restriction on $s(M)$, which is actually a function $\phi = f \circ s$ on $\mathbb{R}^2$:

$$\phi(u, v) := f(0, u, v, 0). \tag{25}$$

Taking (24) and (25) into account, the first condition in (18) (the "complex part") can be written in the form

$$e^{i\theta}\left(\partial_u + i\partial_v + \pi c_1(u + iv)\right)\phi = 0 \quad \text{or} \quad 2\frac{\partial \phi}{\partial \overline{w}} + \pi c_1 w \phi = 0.$$

The general solution to this equation has the form

$$\phi = \exp\left(-\frac{1}{2}\pi c_1 |w|^2\right) \cdot \psi \tag{26}$$

where ψ is a holomorphic function of the complex variable w.

Let us define a holomorphic line bundle L over M so that the function

$$\phi_0(w) = \exp\left(-\frac{1}{2}\pi c_1 |w|^2\right)$$

is by definition a holomorphic section of L. Since this section is nowhere vanishing, all other holomorphic sections have the form $\phi = \psi \cdot \phi_0$ with ψ holomorphic. $\square$

The action of G in the space $L(G, F, \mathfrak{p})$ is computed, as usual, by the master equation

$$s(u, v) \cdot g(\theta, a, b, c) = \exp(\tau T + \gamma Z) \cdot s(u', v')$$

with given u, v, θ, a, b, c and unknown τ, γ, u', v'. The solution is

$$\tau = \theta, \qquad \gamma = c + \frac{(au + bv)\sin\theta + (bu - av)\cos\theta}{2},$$
$$u' = u\cos\theta + v\sin\theta + a, \qquad v' = -u\sin\theta + v\cos\theta + b.$$

Hence, the action has the form

(27)
$$(\pi(g(0, a, b, 0))\phi)(u, v) = e^{\pi i c_1 (bu - av)} \cdot \phi(u + a, v + b),$$
$$(\pi(g(\theta, 0, 0, c))\phi)(u, v) = e^{2\pi i (c_1 c + c_2\theta)} \cdot \phi(u\cos\theta + v\sin\theta, -u\sin\theta + v\cos\theta).$$

The corresponding representation of the Lie algebra $\mathfrak{g}$ in the space of functions ϕ is

$$\begin{aligned}
\pi_*(X) &= \partial_u - \pi i c_1 v;\\
\pi_*(Y) &= \partial_v + \pi i c_1 u;\\
\pi_*(Z) &= 2\pi i c_1\,;\\
\pi_*(T) &= v\,\partial_u - u\partial_v + 2\pi i c_2.
\end{aligned}$$

In terms of the holomorphic functions $\psi = \phi/\phi_0$, it can be rewritten as

$$\begin{aligned}
\pi_*(X) &= \partial_w - \pi c_1 w;\\
\pi_*(Y) &= i\partial_w + \pi i c_1 w;\\
\pi_*(Z) &= 2\pi i c_1;\\
\pi_*(T) &= -iw\,\partial_w + 2\pi i c_2.
\end{aligned}$$

The group representation operators in terms of ψ are

$$(27')\quad \begin{aligned}
(\pi(g(0, a, b, c))\psi)(w) &= e^{2\pi i c_1 c + \pi c_1\left(w(a - ib) - \frac{1}{2}(a^2 + b^2)\right)} \cdot \psi(w + a + ib),\\
(\pi(g(\theta, 0, 0, 0))\psi)(w) &= e^{2\pi i c_2\theta} \cdot \psi(e^{-i\theta} w).
\end{aligned}$$

Now consider the representation space $\mathcal{H}$ more carefully. By definition, it is a completion of $L(G, F, \mathfrak{p})$ with respect to a norm related to a G-invariant inner product. The natural choice for this inner product is

$$(\psi_1, \psi_2) = \int_M \psi_1\overline{\psi_2}\,|\,\phi_0|^2\alpha^*\sigma = \int_{\mathbb{R}^2} \psi_1(w)\overline{\psi_2(w)}\,e^{-\pi c_1|w|^2}\,c_1\,du\wedge dv.$$

It is clear that $\mathcal{H}$ is non-zero exactly when $c_1 > 0$, i.e. when $\mathfrak{p}$ is a Kähler polarization.

As an orthonormal basis in $\mathcal{H}$ we can take the holomorphic functions

$$\psi_n = \frac{(\pi c_1 w)^n}{\sqrt{n!}}, \qquad n \geq 0.$$

Exercise 7. Introduce a Hermitian structure on L so that the representation space $\mathcal{H}$ is isomorphic to the space of all square-integrable holomorphic sections of the bundle L.

Answer. For a holomorphic section ψ define the norm of its value at w by

$$\|\psi(w)\|^2 := |\psi(w)\phi_0(w)|^2.$$

♣

3.3. Representations corresponding to cylindrical orbits.

Here we consider the orbits Ω_r, $r > 0$, which lie in the hyperplane $Z = 0$. According to Rule 3 of the User's Guide, these orbits correspond to unirreps of G that are trivial on the subgroup $\exp(\mathbb{R}\cdot\mathbb{Z})$ of G, so, actually, to unirreps of the quotient group $\widetilde{E}(2)$.

These orbits have a non-trivial fundamental group because the stabilizer $Stab(F)$ of a point $F(t, x, y, 0) \in \Omega_r$ has the form

$$\exp\left(2\pi\mathbb{Z}\cdot T + \mathbb{R}\cdot(xX + yY) + \mathbb{R}\cdot Z\right).$$

Therefore, $\pi_1(\Omega_r) \cong Stab(F)/Stab^0(F) \cong \mathbb{Z}$. We see that a 1-dimensional family of rigged orbits $\Omega_r(\tau)$ corresponds to an orbit Ω_r, labelled by $\tau \in \widehat{\mathbb{Z}} \cong \mathbb{R}/\mathbb{Z}$.

More precisely, a rigged orbit $\Omega_r(\tau) \in \mathcal{O}_{rigg}$ consists of pairs $(F, \chi_{r,\tau})$ where $F \in \Omega_r$ and the character $\chi_{r,\tau} \in \widehat{H}$ is given by

$$\chi_{r,\tau}(\exp\left(2\pi n\cdot T + a\cdot(xX + yY) + c\cdot Z\right)) = e^{2\pi i\,(n\tau + ar)}. \tag{28}$$

Exercise 8. Show that for any $F \in \Omega_r$ there exists a unique real polarization $\mathfrak{h} = \mathbb{R}\cdot X \oplus \mathbb{R}\cdot Y \oplus \mathbb{R}\cdot Z$ and two complex polarizations $\mathfrak{p}_\pm = \mathbb{C}\cdot T \oplus \mathbb{C}\cdot(X \pm Y)$.

Hint. Compute the action of $Stab(F)$ in $T_F\Omega_r$ (resp. in $T_F^{\mathbb{C}}\Omega_r$) and check that it has a unique $Stab(F)$-invariant 1-dimensional subspace $\mathbb{R}\cdot\partial_t$ (resp. three $Stab(F)$-invariant 1-dimensional subspaces $\mathbb{C}\cdot(\partial_u \pm i\partial_v)$ and $\mathbb{C}\cdot\partial_t$). ♣

The **modified Rule 2** attaches the unirrep $\pi_{\Omega_r(\tau)} = \operatorname{Ind}_H^G \chi_{r,\tau}$ to $\Omega_r(\tau)$.

Exercise 9. Derive the following explicit realization of the representation $\pi_{\Omega_r(\tau)}$ in $L^2([0,\, 2\pi),\, d\alpha)$:

$$(29)\qquad \begin{aligned} (\pi_{\Omega_r(\tau)}(g(0,\, a,\, b,\, c))f)(\alpha) &= e^{2\pi i r\,(a\cos\alpha + b\sin\alpha)} f(\alpha), \\ (\pi_{\Omega_r(\tau)}(g(\theta,\, 0,\, 0,\, 0))f)(\alpha) &= e^{2\pi i n\tau} f(\theta_1) \end{aligned}$$

where $n \in \mathbb{Z}$ and $\theta_1 \in [0,\, 2\pi)$ are defined by $\alpha + \theta = \theta_1 + 2\pi n$.

Hint. Use the master equation. ♣

The corresponding representation $\pi_* := (\pi_{\Omega_r(\tau)})_*$ of $\mathfrak{g}$ has the form

$$\pi_*(T) = \partial_\alpha, \quad \pi_*(X) = 2\pi i r\cos\alpha, \quad \pi_*(Y) = 2\pi i r\sin\alpha, \quad \pi_*(Z) = 0.$$

Remark 3. At first glance π_* seems to be independent of the parameter τ. The point is that the operator $A = i\partial_\alpha$ with the domain $\mathcal{A}(0,\, 2\pi) \subset L^2([0,\, 2\pi),\, d\alpha)$ is symmetric but not essentially self-adjoint. It has several self-adjoint extensions A_τ labelled just by the parameter $\tau \in \mathbb{R}/\mathbb{Z}$.

Namely, the domain of A_τ consists of all continuous functions f on $[0,\, 2\pi]$ that have a square-integrable generalized derivative f' and satisfy the boundary condition $f(2\pi) = e^{2\pi i\tau} f(0)$. ♡

The same modified Rule 2 attaches the unirrep $\widetilde{\pi}_{r,\tau}$ to Ω_r, which is holomorphically induced from a subalgebra $\mathfrak{p}_+$. We leave it to the reader to write the explicit formula for $\widetilde{\pi}_{r,\tau}$ acting in the space of holomorphic functions on $\mathbb{C}$ and to check the equivalence of $\widetilde{\pi}_{r,\tau}$ and $\pi_{r,\tau}$.

4. Amendments to other rules

4.1. Rules 3–5.

Here we discuss briefly the **amendments to Rules 3, 4, and 5** suggested by Shchepochkina.

Let H be a closed subgroup of G. We say that a rigged orbit $\Omega' \in \mathcal{O}_{rigg}(H)$ **lies under** a rigged orbit $\Omega \in \mathcal{O}(G)$ (or, equivalently, Ω **lies over** Ω') if there exists a rigged moment $(F, \chi) \in \Omega$ and a moment $(F', \chi') \in \Omega'$ such that the following conditions are satisfied:

$$(30)\qquad \text{a) } p(F) = F', \qquad\qquad \text{b) } \chi = \chi' \text{ on } H \cap Stab(F).$$

We also define the **sum of rigged orbits** Ω_1 and Ω_2 as the set of all $(F,\, \chi)$ for which there exist $(F_i,\, \chi_i) \in \Omega_i,\ i = 1, 2$, such that

$$(31)\qquad F = F_1 + F_2 \quad \text{and} \quad \chi = \chi_1\chi_2 \text{ on } Stab(F_1) \cap Stab(F_2).$$

Theorem 4 (Shchepochkina). *Let G be a connected and simply connected tame solvable Lie group, and let H be a closed tame subgroup. Then*

1. *The spectrum of* $\mathrm{Ind}_H^G \rho_{\Omega'}$ *consists of those* π_Ω *for which* Ω *lies over* Ω'.

2. *The spectrum of* $\mathrm{Res}_H^G \pi_\Omega$ *consists of those* $\rho_{\Omega'}$ *for which* Ω' *lies under* Ω.

3. *The spectrum of* $\pi_{\Omega_1} \otimes \pi_{\Omega_2}$ *consists of those* π_Ω *for which* Ω *is contained in* $\Omega_1 + \Omega_2$.

The proof, as before, is based on the induction on the dimension of G and uses the following result which is interesting on its own right.

Let G be a connected and simply connected solvable Lie group, H its closed subgroup, and K the maximal connected normal subgroup of G that is contained in H. Then the quotient group G/K is called a **basic group**.

Lemma 3 (Shchepochkina). *Any basic group coincides with the simply connected covering of one of the following groups:*

1. *The only non-commutative* 2*-dimensional Lie group* $G_2 = \mathrm{Aff}\,(1, \mathbb{R})$.

2. *The one-parametric family of* 3*-dimensional groups* $G_3(\gamma)$ *in* $\mathrm{Aff}\,(1, \mathbb{C})$ *acting on* $\mathbb{C}$ *by translations and by multiplications by* $e^{\gamma}t$, $\gamma \in \mathbb{C}\backslash\mathbb{R}$.

3. *The* 4*-dimensional group* $G_4 = \mathrm{Aff}(1, \mathbb{C})$. □

Exercise 10. (We use the notation of Section 3.) Find the spectrum of the following representations:

a) $\mathrm{Res}_H^G \pi_{c_1,c_2}$ where $H = \exp(\mathbb{R} \cdot X \oplus \mathbb{R} \cdot Y \oplus \mathbb{R} \cdot Z)$.

b) $\mathrm{Ind}_{C^0}^G 1$ where $C^0 = \exp(\mathbb{R}\cdot Z)$ and 1 denotes the trivial representation.

c) $\mathrm{Ind}_{\exp(\mathbb{R}\cdot T)}^G 1$.

d) $\pi_{r_1}(\tau_1) \otimes \pi_{r_2}(\tau_2)$.

Hint. Use the modified Rules 3–5.

Answer: a) The irreducible representation with $\hbar = c_1$.

b) The continuous sum of all $\pi_r(\tau)$, $r > 0$, $\tau \in \mathbb{R}/\mathbb{Z}$.

c) The continuous sum of all π_{c_1,c_2} with $c_1 c_2 > 0$.

d) The continuous sum of all $\pi_r(\tau)$ with $|r_1 - r_2| \leq r \leq r_1 + r_2$ and $\tau = \tau_1 + \tau_2$. ♣

4.2. Rules 6, 7, and 10.

The **amendment to Rule 6** looks as follows. Consider an orbit $\Omega \in \mathcal{O}(G)$ and let $\{\Omega_\chi\}_{\chi\in\widehat{\pi_1(\Omega)}}$ be the set of all rigged orbits in $\Pi^{-1}(\Omega)$. Then

$$\int_{\widehat{\pi_1(\Omega)}} \operatorname{tr} T_{\Omega_\chi}(\exp X) d\chi = \frac{1}{p(X)} \int_\Omega e^{2\pi i\langle F, X\rangle + \sigma}.$$

In other words, the Fourier transform of the canonical measure on the orbit Ω is equal to the average of all characters of the unirreps corresponding to the rigged orbits from $\Pi^{-1}(\Omega)$.

It would be interesting to find the formula for the character of an individual unirrep T_{Ω_χ}. The example in Section 3.3 could be instructive.

The **modified Rule 8** needs the following natural refinement: for a rigged orbit Ω passing through the point $(F, \chi) \in \mathfrak{g}^*_{rigg}$, we denote by $-\Omega$ the rigged orbit passing through the point $(-F, \overline{\chi})$. (Note that $Stab(F) = Stab(-F)$.)

The **modified Rule 10** for a unimodular group G follows from the above amendment to Rule 6. Namely, the Plancherel measure μ on $\widehat{G} \cong \mathcal{O}_{rigg}(G)$ is concentrated on generic rigged orbits; it induces the normalized Haar measure on each fiber of the projection $\Pi : \mathcal{O}_{rigg}(G) \to \mathcal{O}(G)$ and $\Pi(\mu)$ is the conditional measure on $\mathcal{O}(G)$ corresponding to the Lebesgue measure on $\mathfrak{g}^*$.

Example 5. Consider the group $\widetilde{E}(2)$, the simply connected covering of $E(2)$. It is isomorphic to the quotient group G/C^0 where G is the diamond group from Section 3 and $C^0 = \exp(\mathbb{R} \cdot Z)$ is the connected component of the center in G. So, the space $\widehat{\widetilde{E}(2)}$ is just a subspace of $\widehat{G}$ which consists of rigged cylindrical and one-point orbits.

Exercise 11. Compute the canonical measure on the orbit Ω_r.

Answer: $\sigma = \frac{x\,dy - y\,dx}{x^2+y^2} \wedge dt.$ ♣

By comparing this formula with the Lebesgue measure $dx \wedge dy \wedge dt$ on the momentum space, we conclude that the Plancherel measure on the set of rigged orbits $\Omega_r(\tau)$ is given by

$$\mu = r dr \wedge d\tau.$$

♢

Remark 4. If we consider the group $M(2)$ itself, then the dual space $\widehat{M(2)}$ will be a subset of $\widehat{\widetilde{M}(2)}$. Namely, only representations $\pi_r(0)$ and 1-dimensional representations corresponding to integer one-point orbits will be single-valued on the group. ♡

Chapter 5

Compact Lie Groups

The case of compact Lie groups is the oldest and the most developed part of the representation theory of Lie groups. The simplest compact Lie group is a circle S^1, also known as the 1-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The representation theory of this group has already been used (under the name **Fourier analysis**) for two centuries.

More generally, **commutative Fourier analysis** deals with abelian Lie groups, most of which are direct products of the form $\mathbb{R}^n \times \mathbb{T}^m \times \mathbb{Z}^l \times F$ where F is a finite abelian group.

We leave aside this part of the theory because the orbit method has nothing to add to it. But, certainly, we shall use the basic notions and results of commutative Fourier analysis in our exposition.

Our main object will be the representation theory of non-abelian connected compact Lie groups. Such a group K can have a non-trivial center C. Since the coadjoint representation is trivial on the center, the picture of coadjoint orbits does not change if we replace K by the adjoint group $K_1 = K/C$, or by its simply connected covering $\widetilde{K_1}$, or by some intermediate group between K_1 and $\widetilde{K_1}$. We shall see soon that there are only finitely many such groups because the center of $\widetilde{K_1}$ is finite. In many situations, including the classification problem, it is convenient to consider compact groups up to local isomorphism.

Therefore, we assume from now on that K is a connected and simply connected compact Lie group with finite center. Then the Lie algebra $\mathfrak{k} = \mathrm{Lie}(K)$ is semisimple and we can use the classification theorems from Appendix III.2.3 to get the full list of groups in question.

It is known that any simply connected compact Lie group is isomorphic to a product of groups with simple Lie algebras. The latter form two families:

1. The **classical compact groups**, which are special unitary groups $SU(n, K)$ over $K = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, i.e. automorphism groups of real, complex, or quaternionic Hilbert spaces of finite dimension. (In the case $K = \mathbb{R}$ the group $SO(n, \mathbb{R})$ has the fundamental group $\mathbb{Z}_2$, so we have to consider the simply connected covering group $\mathrm{Spin}(n, \mathbb{R})$; see below.)

2. The so-called **exceptional** groups $\mathbf{E_6}$, $\mathbf{E_7}$, $\mathbf{E_8}$, $\mathbf{F_4}$, $\mathbf{G_2}$.

For classical compact groups almost all of the basic results have been known since the 1920's and were intensively used in quantum physics, which appeared just at that time.

The exceptional groups, until recently, were considered a curious but useless anomaly in the general picture. But the recent development of quantum field theory showed that these groups may play an essential role in new domains of mathematical physics, such as string theory and mirror symmetry. So, the study of these groups and their representations received renewed attention.

At first sight it is hard to expect that one can find something new in this well-explored domain. Nevertheless, the orbit method gives a new insight into known results and even reveals some new facts.

At the same time the idyllic harmony of the correspondence between unirreps and orbits starts to break down in this case. The most intriguing new circumstance can be described as follows. We saw above that there are essentially two ways to establish the correspondence between orbits and unirreps:

1) via the induction-restriction functors,

2) via character theory.

It turns out that for compact groups these two approaches lead to different results!

It is a temptation to consider this discrepancy as a peculiar manifestation of the uncertainty principle: to a given unirrep we can associate an orbit only with a certain degree of accuracy (of size ρ = half the sum of positive roots – see the details below).

1. Structure of semisimple compact Lie groups

This material can be found in many textbooks (see, e.g., [**Bou**], [**FH**], [**Hu**], [**J**]). Nevertheless, we prefer to give a short exposition here of the necessary information to make the book reasonably self-contained. We shall use the notation and results concerning abstract root systems from Appendix III.3.1.

1.1. Compact and complex semisimple groups.

In Appendix III.3.3 we introduce a root system for a pair $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is a complex semisimple Lie algebra and $\mathfrak{h}$ is its Cartan subalgebra. Now we use the theory of abstract root systems to study compact Lie groups.

We keep the following notation:

K — a connected and simply connected compact Lie group of rank n;

$\mathfrak{k} = \mathrm{Lie}(K)$ — the Lie algebra of K;

T — a maximal torus (maximal abelian subgroup) in K;

$\mathfrak{t}$ — the Lie algebra of T;

C — the center of K, a finite group of order c;

$\mathfrak{g} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$ — the complexification of $\mathfrak{k}$;

G — a connected and simply connected complex Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$;

$\mathfrak{h} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ — the complexification of $\mathfrak{t}$, a Cartan subalgebra of $\mathfrak{g}$;

$H = \exp \mathfrak{h}$ — a Cartan subgroup of G;

$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$ — a canonical decomposition of $\mathfrak{g}$;

R — the root system for the pair $(\mathfrak{g}, \mathfrak{h})$;

R_+, Π — the set of positive roots and the set of simple roots, respectively;

$(X, Y)_K$ — the Killing form on $\mathfrak{g}$;

$\check{\lambda}$ (for $\lambda \in \mathfrak{h}^*$) — the element of $\mathfrak{h}$ such that $(\check{\lambda}, H)_K = \lambda(H)$ for all $H \in \mathfrak{h}$;

$(\lambda, \mu) := (\check{\lambda}, \check{\mu})_K$ — the dual form on $\mathfrak{g}^*$ restricted to $\mathfrak{h}^*$.

Recall that the Lie algebra $\mathfrak{k}$ is the compact real form of $\mathfrak{g}$ and is spanned by the elements

$$\frac{X_\alpha - X_{-\alpha}}{2}, \quad \frac{X_\alpha + X_{-\alpha}}{2i} \quad \text{for all } \alpha \in R_+, \quad \text{and} \quad i\check{\alpha}, \text{ for } \alpha \in R. \tag{1}$$

Here $\{X_\alpha,\ \alpha \in R\}$ is a part of the canonical **Chevalley basis** in $\mathfrak{g}$ with the following properties:

$$\begin{array}{lll} & \text{(i)} & [H, X_\alpha] = \alpha(H)X_\alpha \quad \text{for } H \in \mathfrak{h}; \\ & \text{(ii)} & (X_\alpha, X_\beta)_K = \delta_{\alpha,-\beta}; \\ (2) & \text{(iii)} & [X_\alpha, X_{-\alpha}] = \check{\alpha}; \\ & \text{(iv)} & [X_\alpha, X_\beta] = N_{\alpha,\beta}X_{\alpha+\beta} \quad \text{if } \alpha + \beta \in \mathbb{R} \end{array}$$

where $N_{\alpha,\beta}$ are non-zero integers satisfying $N_{-\alpha,-\beta} = -N_{\alpha,\beta} = N_{\beta,\alpha}$.

Lemma 1. a) *The Killing form on $\mathfrak{g}$ and its restriction to $\mathfrak{h}$ are non-degenerate.*

b) *The restriction of the Killing form on $\mathfrak{k}$ is negative definite.*

Proof. Any real representation of a compact group K is equivalent to an orthogonal one. Therefore, operators $\operatorname{ad} X$ for $X \in \mathfrak{k}$ are given by real and skew-symmetric matrices. Hence, $\operatorname{tr}(\operatorname{ad} X^2) = -\operatorname{tr}\big(\operatorname{ad} X \cdot (\operatorname{ad} X)^t\big) \leq 0$.

The equality takes place only when $\operatorname{ad} X = 0$. Since $\mathfrak{k}$ has zero center, it means that $X = 0$. So, we have proved b).

The first part of a) follows from b). The second part of a) follows from the relation $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\alpha + \beta = 0$. □

Remark 1. On a simple Lie algebra the Ad-invariant form is unique up to a numerical factor, hence any such form is proportional to the Killing form.

In particular, for any linear representation π of $\mathfrak{g}$ there is a constant $c_2(\pi)$ such that

$$\operatorname{tr}\big(\pi(X)\pi(Y)\big) = c_2(\pi) \cdot (X, Y)_K. \tag{3}$$

The constant $c_2(\pi)$ is called the **second index** of the representation π (see [**MPR**] for the tables of the second index).

The etymology of this term comes from the following consideration. Let $\{X_i\}$ be any basis in $\mathfrak{g}$, and let $\{X^i\}$ be the dual basis with respect to the Killing form. The element

$$\Delta = \sum_i X_i X^i \in U(\mathfrak{g}) \tag{4}$$

does not depend on the choice of basis. It is called the **quadratic Casimir element**.

One can easily compute that for any unirrep π

$$c_2(\pi) = \frac{I_\pi(\Delta) \cdot \dim\, \pi}{\dim\, \mathfrak{g}} = \frac{1}{\dim \mathfrak{g}} \Delta \chi_\pi(e) \tag{5}$$

where I_π is the infinitesimal character of π. According to formula (14) below, $I_{\pi_\lambda}(\Delta) = |\lambda + \rho|^2 - |\rho|^2$.

There is one special representation π_ρ, $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$, for which the structure of weights is well known (cf. the character formula (12) below). Namely, let r be the number of positive roots. Then $d_\rho := \dim\, \pi_\rho = 2^r$ and there are 2^r weights of the form $\frac{1}{2}\sum_{\alpha \in R_+} \pm\alpha$ where all possible choices of signs occur.

We mention two remarkable corollaries of the formula for $c_2(\pi)$:

$$c_2\big(\pi_{(k-1)\rho}\big) = \frac{k^{r+2} - k^r}{24} \qquad \text{and} \qquad |\rho|^2 = \frac{\dim \mathfrak{g}}{24}.$$

♡

Now let K be a simply connected compact Lie group, and let G be the corresponding simply connected complex Lie group. From the general theory of Lie groups (see, e.g., [**Hel**]) we know that K is a maximal compact subgroup in G and the space $M = G/K$ is contractible.

In the space $\mathfrak{t}^*$, dual to the Lie algebra of a maximal torus $T \subset K$, we have two lattices: $i \cdot Q \subset i \cdot P \subset \mathfrak{t}^*$. We want to describe here the representation-theoretic meaning of these lattices. For this we recall the well-known bijection between equivalence classes of finite-dimensional holomorphic representations of G and equivalence classes of unitary representations of K.

Starting from any unitary representation (π, V) of K we can define successively a representation (π_*, V) of the real Lie algebra $\mathfrak{k} = \mathrm{Lie}(K)$, then the complex-linear representation $(\pi_*^{\mathbb{C}}, V)$ of $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$, and finally the holomorphic representation $(\pi^{\mathbb{C}}, V)$ of G so that the following diagram is commutative (cf. Theorem 2 from Appendix III.1.3):

$$\begin{array}{ccccc}
\mathfrak{k} & \xrightarrow{i_*} & \mathfrak{g} & \xrightarrow{\pi_*^{\mathbb{C}}} & \mathrm{End}(V) \\
{\scriptstyle\exp}\downarrow & & {\scriptstyle\exp}\downarrow & & {\scriptstyle\exp}\downarrow \\
K & \xrightarrow[i]{} & G & \xrightarrow[\pi^{\mathbb{C}}]{} & \mathrm{Aut}(V).
\end{array}$$

Conversely, any holomorphic finite-dimensional representation of G, being restricted to K, is equivalent to a unitary representation of K.

Recall also that a weight λ of a linear representation (π, V) of G is a linear functional on the Cartan subalgebra $\mathfrak{h}$ such that for some non-zero vector $v \in V$ we have

$$\pi_*(X)v = \lambda(X) \cdot v \qquad \text{for any } X \in \mathfrak{h}. \tag{6}$$

This functional takes pure imaginary values on $\mathfrak{t}$, hence can be considered as an element of $i\mathfrak{t}^*$.

We temporarily denote by P' the set of all weights of all holomorphic finite-dimensional representations of G. It is a subgroup of $i\mathfrak{t}^*$, because if $\lambda \in Wt(\pi_1)$ and $\mu \in Wt(\pi_2)$, then $\lambda + \mu \in Wt(\pi_1 \otimes \pi_2)$ and $-\lambda \in Wt(\pi_1^*)$. We shall see in a moment that P' actually coincides with the weight lattice $P \subset i\mathfrak{t}^*$ defined in terms of an abstract root system.

For $\lambda \in \mathfrak{h}^*$ let the symbol e^λ denote the function on H given by

$$e^\lambda(\exp X) = e^{\lambda(X)} \quad \text{for } X \in \mathfrak{h}. \tag{7}$$

It is not evident (and is not true in general!) that this function is well defined. Indeed, an element $h \in H$ can have different "logarithms" X (i.e. elements $X \in \mathfrak{h}$ such that $h = \exp X$).

However, from (6) we conclude that for any $\lambda \in Wt(\pi, V)$ and the appropriate $v \in V$ we have

$$\pi(\exp X)v = e^{\pi_*(X)} \cdot v = e^{\lambda(X)} \cdot v = e^\lambda(\exp X) \cdot v. \tag{8}$$

Hence, for $\lambda \in P'$ the function e^λ is well defined.

To go further we recall the notion of a **dual lattice**. Let V be a finite-dimensional real or complex vector space, and let V^* be its dual space. To any lattice L in V^* the dual lattice L^* in V is defined by

$$v \in L^* \qquad \Longleftrightarrow \qquad f(v) \in \mathbb{Z} \quad \text{for all } f \in L.$$

It is clear that this notion is self-dual: if we consider V as the dual space to V^*, then L will be the dual lattice to L^*.

Proposition 1. *Let $P^* \subset Q^* \subset i\mathfrak{t}$ be the dual lattices to $Q \subset P \subset i\mathfrak{t}^*$. Then*

a) *the following relations hold:*

$$2\pi i\,(P')^* = \mathfrak{h} \cap \exp^{-1}(e), \qquad 2\pi i\, Q^* = \mathfrak{h} \cap \exp^{-1}(C); \tag{9}$$

b) *the two notions of a weight coincide, i.e. $P' = P$.*

Corollary. *The order c of the center C of K (which is also the center of G) is equal to $\#(P/Q) = \det A$ where A is the Cartan matrix of the root system corresponding to K.*

Using the result of Exercise 9 from Appendix III.3.1, we get the following values for the order of the center of all simply connected compact Lie groups with simple Lie algebras:

$c\big(SU(n, \mathbb{C})\big) = n + 1, \quad c\big(\mathrm{Spin}(2n+1, \mathbb{R})\big) = 2, \quad c\big(SU(n, \mathbb{H})\big) = 2,$

$c\big(\mathrm{Spin}(2n, \mathbb{R})\big) = 4, \quad c(\mathbf{E}_n) = 9 - n.$

The same method gives $c(\mathbf{F_4}) = c(\mathbf{G_2}) = 1$.

Proof of Proposition 1. We start with the first of the relations in (9). For any $X \in \mathfrak{h} \cap \exp^{-1}(e)$ we have

$$1 = e^{\lambda}(\exp X) = e^{\lambda(X)} \quad \text{or} \quad \lambda(X) \in 2\pi i\mathbb{Z}.$$

Therefore $\mathfrak{h} \cap \exp^{-1}(e) \subset 2\pi i\,(P')^*$.

To prove the inverse inclusion, we use the fact that the group G has a faithful finite-dimensional representation. It follows that the collection of all functions e^{λ}, $\lambda \in P'$, separates points of H. Hence, if $X \in (P')^*$, then $e^{\lambda}(\exp 2\pi iX) = e^{2\pi i\lambda(X)} = 1$ for all $\lambda \in P'$ and consequently $\exp(2\pi iX) = e$.

The second relation in (9) is proved by the same argument, but instead of all representations we consider only the adjoint one. The functions e^{λ}, $\lambda \in Q$, do not separate points h_1 and h_2 from H iff $\mathrm{Ad}\, h_1 = \mathrm{Ad}\, h_2$, i.e. $h_1h_2^{-1} \in C$. Therefore, $\exp X \in C$ is equivalent to $\lambda(X) \in 2\pi i\mathbb{Z}$ for all $\lambda \in Q$, i.e. $X \in 2\pi iQ^*$.

Let us now prove b). We use

Lemma 2. *For the simple 3-dimensional Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ we have $P' = P$.*

Proof of the lemma. It is well known (see Appendix III.3.2) that irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ can be enumerated by non-negative integers so that π_n is the n-th symmetric power of the standard (tautological) representation π_1.

Choose the standard basis E, F, H in $\mathfrak{sl}(2, \mathbb{C})$ and put $\mathfrak{h} = \mathbb{C} \cdot H$, $\mathfrak{g}_{\alpha} = \mathbb{C} \cdot E$, $\mathfrak{g}_{-\alpha} = \mathbb{C} \cdot F$. Then in an appropriate basis

$$\pi_n(H) = \begin{pmatrix} n & 0 & 0 & \dots & 0 \\ 0 & n-2 & 0 & \dots & 0 \\ 0 & 0 & n-4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -n \end{pmatrix}.$$

Therefore the evaluation at H identifies the lattice P' with $\mathbb{Z}$. Since the adjoint representation is equivalent to π_2, the root lattice Q is identified with $2\mathbb{Z}$.

The weight lattice P for the root system $\mathbf{A_1}$ is generated by the fundamental weight $\omega = \frac{1}{2}\alpha$. Since $\omega(H) = \frac{1}{2}\alpha(H) = 1$, we see that $P = P'$. □

We return to the general case. Let $\lambda \in P'$. Consider the corresponding representation (π, V) of G and restrict π_* to the subalgebra $\mathfrak{g}(\alpha) \subset \mathfrak{g}$ spanned by $X_{\pm\alpha}$ and H_{α}. Then $\lambda\,\big|_{\mathfrak{g}(\alpha)}$ will be a weight of a representation

of $\mathfrak{g}(\alpha)$. But the expression $\frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$ does not depend on the choice of an Ad-invariant form on $\mathfrak{g}(\alpha)$. Hence, it is an integer and $P' \subset P$.

To prove that $P \subset P'$ it is enough to show that all fundamental weights belong to P'. For this the explicit construction is usually used, which shows that all fundamental weights are highest weights of some irreducible representations of G. This is rather easy for classical groups (see, e.g., [**FH**], [**Hu**], [**Zh1**] and the examples below) and is more involved for exceptional groups. Another way is to use E. Cartan's theorem (Theorem 3 below). □

For future use we define the **canonical coordinates** $(t_1, \dots, t_n)$ on H by

$$t_k(\exp X) = e^{\langle \omega_k, X\rangle}. \tag{10}$$

These coordinates take non-zero complex values. Thus, they identify the Cartan subgroup $H \subset G$ with the subset $(\mathbb{C}^\times)^n \subset \mathbb{C}^n$. Under this identification the maximal torus $T \subset K$ goes to the subset $\mathbb{T}^n \subset \mathbb{C}^n$, which consists of points whose coordinates have absolute value 1.

At the same time the lattice P is identified with $\mathbb{Z}^n$ so that the function e^λ has the form

$$e^\lambda(h) = t_1^{k_1}(h)\cdots t_n^{k_n}(h) \qquad \text{for } \lambda = (k_1, \dots, k_n). \tag{11}$$

Example 1. Let $K = SU(N, \mathbb{C})$, $N = n+1$. Then $\mathfrak{k} = \mathfrak{su}(N, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{k}_\mathbb{C}$ is the simple complex Lie algebra of type $\mathbf{A}_n$. The maximal torus T is the diagonal subgroup of K and a general element of T with the canonical coordinates $(t_1, \dots, t_n)$ has the form

$$t = \begin{pmatrix} t_1 & 0 & 0 & \dots & 0 \\ 0 & t_2t_1^{-1} & 0 & \dots & 0 \\ 0 & 0 & t_3t_2^{-1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & t_nt_{n-1}^{-1} & 0 \\ \dots & \dots & \dots & 0 & t_n^{-1} \end{pmatrix}.$$

The central elements have coordinates $t_k = \epsilon^k$, where $\epsilon \in \mathbb{C}$ satisfies $\epsilon^{n+1} = 1$.

The first fundamental representation π_1 of K is just the defining representation: $\pi_1(g) = g$. All other fundamental representations are the exterior powers of it: $\pi_k = \bigwedge^k \pi_1$. Indeed, the highest weight vector in $\bigwedge^k \mathbb{C}^n$ is $e_1 \wedge \dots \wedge e_k$ and the corresponding function e^λ is given by $e^\lambda(t) = t_k$. Thus, $\lambda = \omega_k$, the k-th fundamental weight. ♢

In conclusion we collect the main facts here about the representation theory for a compact simply connected Lie group K. In the formulae below ρ denotes the half sum of positive roots.

Theorem 1. a) *Any unirrep π of K is finite dimensional and can be uniquely extended to a holomorphic irreducible representation of the simply connected complex Lie group G with* $\mathrm{Lie}\,(G) = \mathrm{Lie}\,(K)_{\mathbb{C}}$.

b) *A unirrep π is characterized up to equivalence by its highest weight λ, which can be any dominant weight, i.e. a linear combination of fundamental weights with integral non-negative coefficients.*

c) *The character of the unirrep π_λ with the highest weight λ is given by the Weyl formula*

$$\chi_\lambda(t) = \frac{\sum_{w\in W}(-1)^{l(w)}e^{w\cdot(\lambda+\rho)}(t)}{\sum_{w\in W}(-1)^{l(w)}e^{w\cdot(\rho)}(t)} \quad \text{for } t \in T. \tag{12}$$

d) *The dimension of π_λ is given by*

$$d_\lambda = \prod_{\alpha\in R_+} \frac{(\lambda+\rho,\,\alpha)}{(\rho,\,\alpha)}. \tag{13}$$

e) *The infinitesimal character of π_λ takes the value*

$$I_\lambda(\Delta_2) = (\lambda + 2\rho,\,\lambda) = |\lambda+\rho|^2 - |\rho|^2 \tag{14}$$

on the quadratic Casimir element Δ_2.

f) *the multiplicity of the weight μ in the unirrep π_λ is*

$$m_\lambda(\mu) = \sum_{w\in W}(-1)^{l(w)}P\big(w\cdot(\lambda+\rho) - (\mu+\rho)\big) \tag{15}$$

where P is the **Kostant partition function** *on the root lattice defined as the number of ways to write λ as a sum of positive roots. It has the generating function*

$$\sum_{\lambda\in Q} P(\lambda)e^\lambda = \prod_{\alpha\in R_+}(1-e^\alpha)^{-1}. \tag{16}$$

□

To these results we add the Weyl integration formula. For a central function f, i.e. constant on the conjugacy classes in K, it allows us to reduce the integral over the whole group K with the normalized Haar measure dk to the integral over the maximal torus T with the normalized Haar measure dt:

$$\int_K f(g)dg = \frac{1}{|W|}\int_T f(t)|\Delta(t)|^2 dt \tag{17}$$

where

(18)

$$\Delta(t) = \sum_{w\in W}(-1)^{l(w)}e^{w\cdot\rho}(t) = e^{-\rho}\prod_{\alpha\in R_+}(e^\alpha(t)-1) = \prod_{\alpha\in R_+} 2\sinh\frac{\alpha(\log t)}{2}.$$

1.2. Classical and exceptional groups.

In this section we collect some information about exceptional simple complex Lie algebras. It turns out that any such algebra can be constructed as a $(\mathbb{Z}/n\mathbb{Z})$-graded Lie algebra with some classical Lie algebra in the role of $\mathfrak{g}_0$. This allows us to construct matrix realizations of exceptional Lie groups.

Let R be a root system of a simple Lie algebra $\mathfrak{g}$ of rank n, let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a set of simple roots, and let $\psi \in R$ be the maximal root. Let $A = \|A_{ij}\|$ be the Cartan matrix. Define the **extended Dynkin graph** $\widetilde{\Gamma}$ as follows.

The vertices of $\widetilde{\Gamma}$ form the set $\widetilde{\Pi} = \Pi \cup \{\alpha_0\}$ where $\alpha_0 = -\psi$. The vertices α_i and α_j, as usual, are joined by $n_{i,j} = A_{i,j} \cdot A_{j,i}$ edges. The numbers a_k, $0 \le k \le n$, are defined by

$$a_0 = 1 \qquad \text{and} \qquad \sum_{k=0}^{n} a_k \alpha_k = 0. \tag{19}$$

Proposition 2. *Let Γ_k denote the graph obtained from $\widetilde{\Gamma}$ by deleting the k-th vertex together with outgoing edges. Then the Lie algebra $\mathfrak{g}$ can be realized as a $(\mathbb{Z}/a_k\mathbb{Z})$-graded Lie algebra in which $\mathfrak{g}_0$ is a semisimple subalgebra defined by the graph Γ_k.*

Proof. Let $\mathfrak{g}_0$ be the subalgebra in $\mathfrak{g}$ generated by $\{X_{\pm\alpha},\ \alpha \in \Gamma_k\}$. Then, according to Theorem 1 c), $\mathfrak{g}_0$ is a semisimple Lie subalgebra of $\mathfrak{g}$. Relation (19) shows that the root lattice of $\mathfrak{g}_0$ has index a_k in the root lattice of $\mathfrak{g}$. So, $\mathfrak{g}$ as a $\mathfrak{g}_0$-module is $(\mathbb{Z}/a_k\mathbb{Z})$-graded. More precisely, the degree of X_α is equal to $\frac{2(\omega_k, \alpha)}{(\alpha_k, \alpha_k)} \bmod a_k$.

Note also that the highest weight of the $\mathfrak{g}_0$-module of degree 1 is equal to $-\alpha_k$. □

Example 2. Consider the exceptional complex Lie algebra $\mathbf{G_2}$. The extended Dynkin graph and numbers $\{a_k\}$ are shown in Figure 1. (The arrow, as usual, indicates the short root.)

$$\overset{1}{\circ} - - \overset{2}{\circ} \Rrightarrow \overset{3}{\circ}$$

Figure 1. Extended Dynkin graph $\widetilde{\Gamma}$ for $\mathbf{G}_2$.

It follows that $\mathbf{G_2}$ can be defined

a) as a $(\mathbb{Z}/2\mathbb{Z})$-graded Lie algebra $\mathfrak{g}$ with

$$\mathfrak{g}_0 = sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C}), \qquad \mathfrak{g}_1 = V_1 \otimes V_3$$

where V_1 is a standard 2-dimensional representation of the first factor and V_3 is a 4-dimensional representation of the second factor (the symmetric cube of the standard representation);

b) as a $(\mathbb{Z}/3\mathbb{Z})$-graded Lie algebra $\mathfrak{g}$ with

$$\mathfrak{g}_0 = sl(3, \mathbb{C}), \qquad \mathfrak{g}_1 = V, \qquad \mathfrak{g}_{-1} = V^*$$

where V is a standard 3-dimensional $\mathfrak{g}_0$-module which we realize as the space of column vectors and V^* is the dual $\mathfrak{g}_0$-module of row vectors.

The corresponding splittings of the root system are shown in Figure 2.

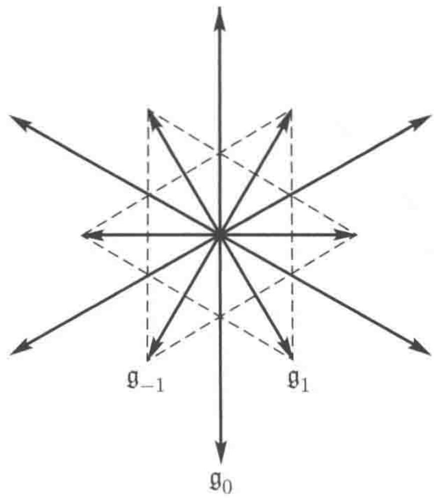

Figure 2. Splittings of the root system for $\mathbf{G}_2$.

We consider in detail the second approach. The commutators of homogeneous elements $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$ are defined as follows.

Let $X, Y \in \mathfrak{g}_0$, $v, v' \in V$, $f, f' \in V^*$. Then

$$[X, Y] = XY - YX; \qquad [X, v] = Xv; \qquad [f, X] = fX;$$
$$[v, f] = vf - \frac{1}{3} fv \cdot 1, \qquad [v, v'] = f(v, v'), \qquad [f, f'] = v(f, f').$$

Here $f(v, v') \in V^*$ and $v(f, f') \in V$ are defined by the relations

$$\langle f(v, v'), v'' \rangle = a \cdot \det | v \quad v' \quad v'' | \quad \text{and} \quad \langle f'', v(f, f') \rangle = b \cdot \det \begin{vmatrix} f \\ f' \\ f'' \end{vmatrix}$$

where a and b are appropriate constants. The Jacobi identity implies that $3ab = 4$, so we come to the matrix realization of $\mathbf{G_2}$ by 7×7 complex

matrices of the form

$$\mathcal{X} = \begin{pmatrix} X & v & A(\check{f}) \\ f & 0 & -\check{v} \\ A(v) & -\check{f} & -\check{X} \end{pmatrix} \tag{20}$$

where A is the linear map from V to $\mathfrak{so}(3, \mathbb{C})$ given by

$$A(v) = \frac{1}{\sqrt{3}} \begin{pmatrix} v^2 & -v^1 & 0 \\ -v^3 & 0 & v^1 \\ 0 & v^3 & -v^2 \end{pmatrix} \quad \text{for } v = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}.$$

The compact real form of $\mathbf{G_2}$ comes out if we assume X to be a skew-Hermitian matrix and $v = -f^t$. To finish with the description of $\mathbf{G_2}$, we mention that the 7-dimensional representation (20) identifies the compact real form of $\mathbf{G_2}$ with the Lie algebra of derivations of the 8-dimensional **octonion algebra** $\mathbb{O}$, restricted to the subspace of pure imaginary elements. The point is that for any matrix $\mathcal{X}$ of the form (20) the linear transformation given by $\exp \mathcal{X}$ preserves some trilinear form $\{a, b, c\}$ in $\mathbb{R}^7$. Since it also preserves a bilinear form $(a, b) = \check{a}b$, we can define a non-associative bilinear operation in $\mathbb{R}^8$:

$$(\alpha, a) * (\beta, b) = (\gamma, c) \quad \text{where } \gamma = \alpha\beta - (a, b) \text{ and } (c, c') = \{a, b, c'\}.$$

This is exactly the octonion algebra. ♡

For the interested reader we show the extended Dynkin graphs and numbers a_k for other exceptional simple Lie algebras in Figure 3.

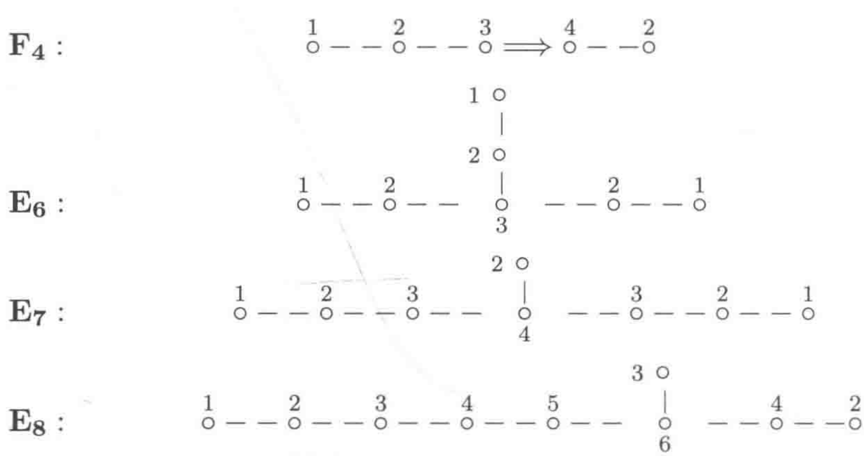

Figure 3. Extended Dynkin graphs.

The reader easily recognizes the rule which governs the distribution of numbers a_k: there is a maximal value $a_j = \max\limits_k a_k$ and all other numbers form an arithmetic progression on any segment of the Dynkin graph starting from $\{j\}$.

Note also that the sum of all numbers a_k is usually denoted by h and is called the **Coxeter number** of the given root system.

Exercise 1.* Describe the graded realizations of all remaining exceptional Lie algebras.

Answer: For $\mathbf{F_4}$:

1) $\mathfrak{g}_0 = \mathbf{B_4}$, $\mathfrak{g}_1 = V_{\omega_4}$;
2) $\mathfrak{g}_0 = \mathbf{C_3} \times \mathbf{A_1}$, $\mathfrak{g}_1 = V_{\omega_3} \otimes V_{\omega_1}$;
3) $\mathfrak{g}_0 = \mathbf{A_2} \times \mathbf{A_2}$, $\mathfrak{g}_1 = V_{\omega_1} \otimes V_{2\omega_1}$, $\mathfrak{g}_{-1} = V_{\omega_2} \otimes V_{2\omega_2}$;
4) $\mathfrak{g}_0 = \mathbf{A_3} \times \mathbf{A_1}$, $\mathfrak{g}_1 = V_{\omega_1} \otimes V_{\omega}$, $\mathfrak{g}_2 = V_{\omega_2} \otimes V_{2\omega}$, $\mathfrak{g}_3 = V_{\omega_3} \otimes V_{\omega}$.

For $\mathbf{E_6}$:

1) $\mathfrak{g}_0 = \mathbf{A_5} \times \mathbf{A_1}$, $\quad \mathfrak{g}_1 = V_{\omega_3} \otimes V_{\omega_1}$;
2) $\mathfrak{g}_0 = \mathbf{A_2} \times \mathbf{A_2} \times \mathbf{A_2}$, $\quad \mathfrak{g}_1 = V_{\omega_1} \otimes V_{\omega_1} \otimes V_{\omega_1}$, $\quad \mathfrak{g}_{-1} = V_{\omega_2} \otimes V_{\omega_2} \otimes V_{\omega_2}$.

For $\mathbf{E_7}$:

1) $\mathfrak{g}_0 = \mathbf{A_7}$, $\quad \mathfrak{g}_1 = V_{\omega_4}$;
2) $\mathfrak{g}_0 = \mathbf{D_6} \times \mathbf{A_1}$, $\mathfrak{g}_1 = V_{\omega_6} \otimes V_{\omega_1}$;
3) $\mathfrak{g}_0 = \mathbf{A_5} \times \mathbf{A_2}$, $\mathfrak{g}_1 = V_{\omega_1} \otimes V_{\omega_2}$, $\mathfrak{g}_{-1} = V_{\omega_2} \otimes V_{\omega_4}$;
4) $\mathfrak{g}_0 = \mathbf{A_3} \times \mathbf{A_3} \times \mathbf{A_1}$, $\mathfrak{g}_1 = V_{\omega_1} \otimes V_{\omega_1} \otimes V_{\omega}$, $\mathfrak{g}_2 = V_{\omega_2} \otimes V_{\omega_2} \otimes V_0$, $\mathfrak{g}_3 = V_{\omega_3} \otimes V_{\omega_3} \otimes V_{\omega}$.

For $\mathbf{E_8}$:

1) $\mathfrak{g}_0 = \mathbf{A_3} \times \mathbf{D_5}$, $\mathfrak{g}_1 = V_{\omega_1} \otimes V_{\omega_5}$, $\mathfrak{g}_2 = V_{\omega_1} \otimes V_{\omega_5}$, $\mathfrak{g}_3 = V_{\omega_1} \otimes V_{\omega_5}$;
2) $\mathfrak{g}_0 = \mathbf{A_4} \times \mathbf{A_4}$, $\mathfrak{g}_1 = V_{\omega_1} \otimes V_{\omega_2}$, $\mathfrak{g}_2 = V_{\omega_1} \otimes V_{\omega_3}$, $\mathfrak{g}_3 = V_{\omega_4} \otimes V_{\omega_2}$, $\mathfrak{g}_4 = V_{\omega_4} \otimes V_{\omega_3}$;
3) $\mathfrak{g}_0 = \mathbf{A_5} \times \mathbf{A_2} \times \mathbf{A_1}$, $\mathfrak{g}_1 = V_{\omega_1} \otimes V_{\omega_2} \otimes V_{\omega}$, $\mathfrak{g}_2 = V_{\omega_2} \otimes V_{\omega_1} \otimes V_{\omega}$, $\mathfrak{g}_3 = V_{\omega_3} \otimes V_0 \otimes V_{\omega}$, $\mathfrak{g}_4 = V_{\omega_4} \otimes V_{\omega_2} \otimes V_{\omega}$, $\mathfrak{g}_5 = V_{\omega_5} \otimes V_{\omega_1} \otimes V_{\omega}$;
4) $\mathfrak{g}_0 = \mathbf{A_7} \times \mathbf{A_1}$, $\mathfrak{g}_1 = V_{\omega_1} \otimes V_{\omega}$, $\mathfrak{g}_2 = V_{\omega_4} \otimes V_0$, $\mathfrak{g}_3 = V_{\omega_7} \otimes V_{\omega}$;
5) $\mathfrak{g}_0 = \mathbf{D_8}$, $\mathfrak{g}_1 = V_{\omega_8}$;
6) $\mathfrak{g}_0 = \mathbf{A_8}$, $\mathfrak{g}_1 = V_{\omega_3}$, $\mathfrak{g}_{-1} = V_{\omega_6}$.

♣

2. Coadjoint orbits for compact Lie groups

2.1. Geometry of coadjoint orbits.

For a compact Lie group K, in view of the existence of an Ad-invariant non-degenerate bilinear form on $\mathfrak{k} = \mathrm{Lie}\,(K)$, the coadjoint representation is equivalent to the adjoint one.

A well-known theorem about compact transformation groups implies that for a given compact Lie group K there are only finitely many types of (co)adjoint orbits as homogeneous K-manifolds. We give the description of these manifolds here.

It is convenient to identify the functional $F \in \mathfrak{k}^*$ with the element $X_F \in \mathfrak{k}$ via the formula $\langle F_X, Y\rangle = (X, Y)_K$ for all $Y \in \mathfrak{k}$. Then the real vector space $i\mathfrak{t}^*$ spanned by roots will be identified with a subspace of $i\mathfrak{k}$.

Proposition 3. *Let K be any connected compact Lie group. Then*

a) *For any point $X \in \mathfrak{k}$ the stabilizer of X is conjugate to a subgroup S that contains the maximal torus $T \subset K$.*

b) *There are finitely many subgroups intermediate between T and K and each of them is a stabilizer of some point $X \in \mathfrak{t}$.*

c) *For all (co)adjoint orbits of maximal dimension the stabilizer of a point is conjugate to the maximal torus T.*

Proof. We deduce all statements from

Lemma 3. *Let C_+ be the positive Weyl chamber in $i\mathfrak{t}^*$. Any K-orbit in $\mathfrak{k}^* \simeq \mathfrak{k}$ intersects the set $iC_+ \subset \mathfrak{t}$ in exactly one point.*

Proof of the lemma. Let Ω be a K-orbit in $\mathfrak{k}$. Pick a regular element $H \in \mathfrak{t}$ and consider the function $f_H(X) = (X, H)_K$ on Ω. Since Ω is a compact set, this function attains an extremum at some point X_0 which must be a stationary point for f_H. Therefore, for any $X \in \mathfrak{k}$ we have $df_H(X_0)(X) = ([X, X_0], H)_K = 0$. But this implies the equality $([X_0, H], X)_K = 0$ for all $X \in \mathfrak{k}$. Hence, X_0 commutes with H and, consequently, belongs to $\mathfrak{t}$. Thus, the intersection $\Omega \cap \mathfrak{t}$ is non-empty.

Besides, it is a W-invariant subset in $\mathfrak{t}$. We claim that it is a single W-orbit. Indeed, the subalgebra $\mathfrak{t} \subset \mathfrak{k}$ is the centralizer of H in $\mathfrak{k}$. Therefore, if $\operatorname{Ad} k\, H \in \mathfrak{t}$, then $\operatorname{Ad} k\, \mathfrak{t} = \mathfrak{t}$, hence $k \in N_K(T)$.

It remains to show that the W-orbit of any point $\mu \in i\mathfrak{t}^*$ has a unique common point with C_+. Let $\mu \in C_+$ and $w \in W$. Then $w(\mu)$ is separated from μ by $l(w)$ mirrors. Therefore, $w(\mu)$ can belong to C_+ only if it belongs to all these mirrors. But then $w(\mu) = \mu$. □

We return to the proof of Proposition 3. It is clear that the stabilizer of the point $\mu \in \mathfrak{t}$ contains T and coincides with T for regular μ. This proves a). If μ is an arbitrary point of $\mathfrak{t}$, then the Lie algebra of its stabilizer contains $\mathfrak{t}$, hence has the form

$$\mathfrak{k}' = \mathfrak{t} + \sum_{\alpha \in R'} \left(\mathbb{R} \cdot \frac{X_\alpha - X_{-\alpha}}{2} \oplus \mathbb{R} \cdot \frac{X_\alpha + X_{-\alpha}}{2i} \right)$$

where $R' \subset R$ is the intersection of R with the hyperplane $\bigcap \mu^{\perp}$. Evidently, there are only finitely many such subsets in R and we denote them by $R^{(m)}$, $1 \leq m \leq M$.

Let $\mathfrak{k}^{(m)}$ be the subalgebra spanned by $\mathfrak{t}$ and the vectors $\frac{X_\alpha - X_{-\alpha}}{2}$ and $\frac{X_\alpha + X_{-\alpha}}{2i}$, $\alpha \in R^{(m)}$. It is a direct sum of a real semisimple Lie algebra $\mathfrak{k}_1^{(m)} = [\mathfrak{k}^{(m)}, \mathfrak{k}^{(m)}]$ and an abelian subalgebra.

$R^{(m)}$ can be considered as a root system related to the complex semisimple Lie algebra $\mathfrak{g}_1^{(m)} = \mathfrak{k}_1^{(m)} \otimes \mathbb{C}$.

Conversely, any subalgebra $\mathfrak{k}^{(m)}$, $1 \leq m \leq M$, is the centralizer of an element $\mu^{(m)} \in \mathfrak{t}$, and the corresponding subgroup $K^{(m)}$ is the stabilizer of $\mu^{(m)}$ in K. Hence, the K-orbit of $i\mu^{(m)} \in \mathfrak{t}$ has the form $K/K^{(m)}$. □

The space $\mathcal{F} = K/T$ is called the **full flag manifold** for K. The remaining homogeneous spaces $\mathcal{F}^{(m)} = K/K^{(m)}$ are called **degenerate flag manifolds**: they can be obtained from $\mathcal{F}$ by a projection whose fibers are isomorphic to a product of smaller flag manifolds.

Example 3. Let $K = U(n)$, and let $T = T(n)$ be the subgroup of diagonal matrices. The Lie algebra $\mathfrak{k}$ consists of all $n \times n$ skew-Hermitian matrices X. We can define an invariant positive definite bilinear form on $\mathfrak{g}$ by $(X, Y) := -\mathrm{tr}(XY)$.

Part a) of Proposition 3 in this case reduces to a well-known fact that every skew-Hermitian matrix can be reduced to the diagonal form via conjugation by a unitary matrix.

Part b) claims that this diagonal form is unique if we assume that the (pure imaginary) eigenvalues $(i\lambda_1, \ldots, i\lambda_n)$ satisfy the condition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Gathering equal eigenvalues of X in blocks of sizes $n_1, \ldots, n_r$, we see that $Stab(X)$ is conjugate to a block-diagonal subgroup $U_{n_1,\ldots,n_r} \simeq U_{n_1} \times \cdots \times U_{n_r}$. The corresponding orbit has dimension $d = 2\sum_{i<j} n_i n_j$, which varies from 0 to $n(n-1)$. The number of different types of orbits in this case is equal to the number $p(n)$ of all partitions of n.

The **flag manifold** $\mathcal{F}_{n_1,\ldots,n_r}$ can be realized as the manifold of all **filtrations**, which are also called **flags**, of type $(n_1, \ldots, n_r)$:

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_r = \mathbb{C}^n \quad \text{with } \dim V_i/V_{i-1} = n_i$$

(see Figure 4). The most degenerate flag manifolds are **Grassmannians**

$$G_{n,k} = U(n)/U_{k,n-k}.$$

Geometrically they are realized as manifolds of k-dimensional subspaces in $\mathbb{C}^n$. ♢

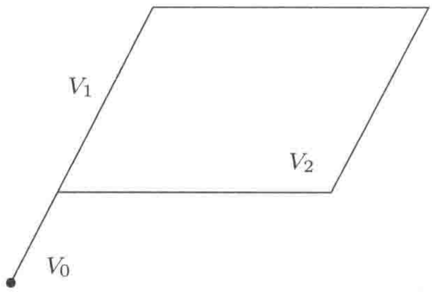

Figure 4. A flag of type (1,2) in $\mathbb{R}^3$.

The flag manifolds have a rich geometric structure, which we describe here.

First, being homogeneous spaces for a compact Lie group K, they admit a K-invariant Riemannian metric.

Second, being coadjoint orbits, they have a canonical K-invariant symplectic structure.

Third, they can be endowed with a K-invariant complex structure, i.e. admit local complex coordinates in which the action of K is holomorphic.

Finally, all three structures can be united into one by saying that flag manifolds are homogeneous Kähler K-manifolds.

We recall (see Appendix II.3.1) that a complex manifold M is called **Kähler** if in every tangent space $T_m(M)$ a Hermitian form $h(x, y)$ is given such that

(i) the real part $g = \operatorname{Re} h$ defines a Riemannian metric on M;

(ii) the imaginary part $\sigma = \operatorname{Im} h$ defines a symplectic structure on M.

Let $z^1, \ldots, z^n$, $z^k = x^k + iy^k$, be a local coordinate system on M. Then the local expressions for h, g, and σ have the form:

$$h = h_{\alpha,\beta}\, dz^\alpha \otimes \overline{dz^\beta}, \quad \text{with } h_{\beta,\alpha} = \overline{h_{\alpha,\beta}} = s_{\alpha,\beta} + ia_{\alpha,\beta}, \quad s_{\alpha,\beta},\ a_{\alpha,\beta} \in \mathbb{R},$$
$$g = s_{\alpha,\beta}(dx^\alpha dx^\beta + dy^\alpha dy^\beta) + 2a_{\alpha,\beta}\, dx^\alpha dy^\beta,$$
$$\sigma = \frac{1}{2} a_{\alpha,\beta}(dx^\alpha \wedge dx^\beta + dy^\alpha \wedge dy^\beta) - s_{\alpha,\beta}\, dx^\alpha \wedge dy^\beta.$$

Lemma 4. *Let M be a complex manifold with local coordinates $z^1, \ldots, z^n$, and let $h = h_{\alpha,\beta} dz^\alpha \overline{dz^\beta}$ be the local expression for a Kähler form on M.*

a) *In a neighborhood of any given point there exists a real-valued function* $\mathbf{K}$ *such that the coefficients of h have the form*

$$h_{\alpha,\beta} = \partial_\alpha \overline{\partial}_\beta \mathbf{K}. \tag{21}$$

b) *The function* $\mathbf{K}$ *is defined modulo the summand of the form* $\operatorname{Re} f$ *where* f *is a holomorphic function. It is called the* **Kähler potential** *of* h.

Sketch of the proof. a) It is convenient to rewrite the symplectic form $\sigma = \operatorname{Im} h$ as $\frac{1}{2} h_{\alpha,\beta} dz^{\alpha} \wedge \overline{dz^{\beta}}$. The condition $d\sigma = 0$ gives

$$\frac{\partial h_{\alpha,\beta}}{\partial z^{\gamma}} dz^{\gamma} \wedge dz^{\alpha} \wedge \overline{dz^{\beta}} + \frac{\partial h_{\alpha,\beta}}{\partial \overline{z^{\gamma}}} dz^{\alpha} \wedge \overline{dz^{\beta}} \wedge \overline{dz^{\gamma}} = 0.$$

Since the summands belong to different types ((2,1) and $(1,2)$, respectively), the equality can be true only if both summands vanish. It follows that

$$\frac{\partial h_{\gamma,\beta}}{\partial z^{\alpha}} = \frac{\partial h_{\alpha,\beta}}{\partial z^{\gamma}} \qquad \text{and} \qquad \frac{\partial h_{\alpha,\gamma}}{\partial \overline{z^{\beta}}} = \frac{\partial h_{\alpha,\beta}}{\partial \overline{z^{\gamma}}}. \tag{22}$$

It is known[1] that the first condition in (22) is sufficient for the existence of functions ϕ_{β} such that locally $h_{\alpha,\beta} = \partial \phi_{\beta} / \partial z^{\alpha}$. Analogously, the second condition in (22) ensures the existence of functions ψ_{α} such that locally $h_{\alpha,\beta} = \partial \psi_{\alpha} / \partial \bar{z}^{\beta}$.

Finally, the equation

$$\frac{\partial \phi_{\beta}}{\partial z^{\alpha}} = \frac{\partial \psi_{\alpha}}{\partial \overline{z^{\beta}}}$$

guarantees that there is a function $\mathbf{K}$ such that locally

$$\phi_{\beta} = \frac{\partial \mathbf{K}}{\partial \overline{z^{\beta}}}, \qquad \psi_{\alpha} = \frac{\partial \mathbf{K}}{\partial z^{\alpha}}.$$

So, we have found a function $\mathbf{K}$ such that the equality (21) holds. Now observe that the complex conjugate function $\overline{\mathbf{K}}$ also satisfies the same equality because the matrix $\|h_{\alpha,\beta}\|$ is Hermitian. Then $\operatorname{Re} \mathbf{K}$ is real and again satisfies (21).

b) This follows from the fact that all real solutions to the system of equations

$$\frac{\partial^2}{\partial z^{\alpha} \partial \overline{z^{\beta}}} f = 0, \qquad \alpha, \beta \in \{1, 2, \ldots, n\},$$

have the form $f = \operatorname{Re} g$ where g is a holomorphic function. □

[1] Actually, we use the Dolbeault-Grothendieck Lemma here (the complex analogue of the Poincaré Lemma): $d'\omega = 0$ locally implies $\omega = d'\theta$. Here d' is the holomorphic part of the differential d. Namely, $d'(f dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}) = \frac{\partial f_{i_1 \ldots i_k}}{\partial z_i} dz^i \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$.

Example 4. Let $K = U(n)$, and let $M = K/U_{n-k,k} = G_{n,k}(\mathbb{C})$, the Grassmann manifold. For $k = 1$ we have $G_{n,1} \simeq \mathbb{P}^{n-1}$ and this example is considered in Appendix II.3.1. The formula obtained there for the Kähler potential can be generalized to all Grassmann manifolds.

First, we want to define a convenient parametrization of $G_{n,k}$. Consider the set $\mathcal{Z}_{n,k}$ of all rectangular complex matrices Z of rank k and size $n \times k$. It is a left $\big(GL(n, \mathbb{C})\big)$-set and a right $\big(GL(k, \mathbb{C})\big)$-set with respect to the ordinary matrix multiplication.

To any $Z \in \mathcal{Z}_{n,k}$ we associate the k-dimensional subspace $[Z]$ in $\mathbb{C}^n$ spanned by the columns of Z. It is clear that the map $\mathcal{Z}_{n,k} \longrightarrow G_{n,k}$: $Z \mapsto [Z]$ is surjective and that $[Z_1] = [Z_2]$ iff Z_1 and Z_2 belong to the same $GL(k, \mathbb{C})$-orbit. Thus, the functions on $G_{n,k}$ are just the $GL(k, \mathbb{C})$-invariant functions on $\mathcal{Z}_{n,k}$.

Now, we define the local coordinate systems on $G_{n,k}$ as follows. Choose a subset $I = \{i_1, i_2, \ldots, i_k\}$ of cardinality k from the set of indices $\{1, 2, \ldots, n\}$, and let $J = \{j_1, j_2, \ldots, j_{n-k}\}$ be the complementary subset. Denote by Z_I the $k \times k$ matrix with elements $a_{ml} = z_{i_m,l}$ and by Z^J the $(n-k) \times k$ matrix with elements $b_{ml} = z_{j_m,l}$. To each I we define the chart U_I on $G_{n,k}$ that contains $[Z]$ for all Z with $\det Z_I \neq 0$. The $(n-k) \times k$ coefficients of the matrix $C^J := Z^J (Z_I)^{-1}$ are by definition the local coordinates on U_I.

Exercise 2. Check that the transition functions are rational functions with non-vanishing denominators.

Hint. Show that for any point $[Z] \in U_I$ one can choose the representative Z such that $Z_I = 1$, $Z^J = C^J$. If $[Z]$ also belongs to another chart $U_{I'}$, compute the matrix $C^{J'}$ in terms of C^J. ♣

Define the function $\mathbf{K}_I$ on U_I by

$$\mathbf{K}_I = -\log \frac{\det (Z^* Z)}{\det (Z_I^* Z_I)}.$$

The differences

$$\mathbf{K}_I - \mathbf{K}_{I'} = \log \frac{\det Z_{I'}}{\det Z_I} + \overline{\log \frac{\det Z_{I'}}{\det Z_I}}$$

are real parts of analytic functions. Therefore, the Kähler form $\partial\bar{\partial}\mathbf{K}_I$ does not depend on the index I and is well defined on $G_{n,k}$.

Note that this form and the corresponding metric and symplectic structure are invariant under the action of $U(n, \mathbb{C})$, but not under the group $GL(n, \mathbb{C})$. ♢

There is a simple explanation for why all (co)adjoint orbits, which are flag manifolds, possess a complex structure. Recall that the group K is a maximal compact subgroup in the simply connected complex Lie group G.

Consider the canonical decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \quad \text{with } \mathfrak{h} = \mathfrak{t}_{\mathbb{C}}.$$

Then $\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$ are maximal solvable subalgebras in $\mathfrak{g}$. They are called **opposite Borel subalgebras**. Let $B_\pm \subset G$ be the corresponding **Borel subgroups**. Often we shall omit the lower index $_+$ in the notation $\mathfrak{n}_+$, $\mathfrak{b}_+$, B_+.

Recall that the subgroup $H = T_{\mathbb{C}}$ is a **Cartan** subgroup of G, i.e. a maximal abelian subgroup which consists of Ad-semisimple elements.

Consider now the complex homogeneous manifold $Y = G/B$.

Lemma 5. *The compact group K acts transitively on Y and the stabilizer of the initial point coincides with $B \cap K = T$.*

Proof. Let $\{H_k,\ 1 \le k \le l,\ X_\alpha,\ \alpha \in R\}$ be a Chevalley basis in $\mathfrak{g}$ where the X_α satisfy commutation relations (2) and the H_k form an orthonormal basis in $i\mathfrak{t}$. The Lie algebra $\mathfrak{k}$ is spanned by elements

$$X_\alpha - X_{-\alpha}, \quad iX_\alpha + iX_{-\alpha} \quad \text{for all } \alpha \in R \quad \text{and} \quad iH_k, \quad 1 \le k \le l.$$

From this we see that

$$\mathfrak{k} + \mathfrak{b} = \mathfrak{g} \qquad \text{and} \qquad \mathfrak{k} \cap \mathfrak{b} = \mathfrak{t}. \tag{23}$$

The first equality in (23) implies that the K-orbit of the initial point $B/B \in G/B$ is open. Since it is compact, it is also closed, hence coincides with $Y = G/B$.

The second equality in (23) shows that locally $K \cap B$ coincides with T. Since T is clearly a maximal compact subgroup in B, this is true globally.□

Thus, we can identify $\mathcal{F}$ with Y and obtain a K-invariant complex structure on the flag manifold $\mathcal{F}$.

For degenerate flag manifolds $\mathcal{F}^{(i)} = K/K^{(i)}$ the situation is analogous; the role of $Y^{(i)}$ is played by $G/P^{(i)}$ where $P^{(i)}$ is a so-called **parabolic** subgroup of G: a minimal complex subgroup in G that contains B and $K^{(i)}$.

Remark 2. A K-invariant complex structure on a flag manifold is not unique. In the case of a full flag manifold $\mathcal{F} = K/T$ all possible complex structures can be described as follows. Let $W = N_K(T)/T$ be the **Weyl**

group corresponding to the pair (K, T). This group acts by automorphisms of $\mathcal{F}$ considered as a homogeneous K-space.[2]

Note that the Weyl group W also coincides with $N_G(H)/H$. Hence, it acts by automorphisms of the homogeneous space $G/H \neq G/B$. But the action of $w \in W$ on G/B is *not holomorphic*!

It turns out that W acts in a simply transitive way on the set $c(\mathcal{F})$ of all different K-invariant complex structures on $\mathcal{F}$. So, $c(\mathcal{F})$ is a principal homogeneous space for W and has cardinality $|W|$.

Note also that there are exactly $|W|$ different Borel subgroups containing H, hence $|W|$ ways to identify $\mathcal{F}$ with Y. ♡

From now on we fix the choice of a complex structure on $\mathcal{F}$ or, in other words, fix a Borel subgroup B such that $B \supset H \supset T$.

Example 5. Let $K = SU(2)$. Then $G = SL(2, \mathbb{C})$, $\mathcal{F} = S^2$, $Y = P^1(\mathbb{C})$ and the two complex structures on S^2 result from the two possible K-equivariant identifications of S^2 with $P^1(\mathbb{C})$. ♢

Exercise 3.* Describe the six different invariant complex structures on the flag manifold $\mathcal{F} = SU(3)/T$.

Hint. The points of the space $Y = SL(3, \mathbb{C})/B$ are geometrically realized by flags $V_1 \subset V_2$ where V_i is an i-dimensional subspace in $\mathbb{C}^3$. We can consider V_1 as a point in the projective plane $P^2(\mathbb{C})$ and V_2, or, more precisely, its annihilator $V_2^{\perp} \subset (\mathbb{C}^3)^*$ as a point in a dual projective plane $P^2(\mathbb{C})^*$. We get a K-equivariant embedding of Y into the product of two dual complex projective planes.

In terms of dual homogeneous coordinates $(y^1 : y^2 : y^3)$ and $(y_1 : y_2 : y_3)$ on these planes the manifold Y is given by the equation

$$y^1 y_1 + y^2 y_2 + y^3 y_3 = 0.$$

Now, a K-equivariant map $\mathcal{F} \to Y$ is completely determined by the image of the initial point $x_0 = T/T \in K/T = \mathcal{F}$. This image must be a fixed point of the T-action on Y. It remains to show that there are exactly six fixed points for T in Y:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where the upper row gives the homogeneous coordinates of a point in $P^2(\mathbb{C})$ while the second row gives the homogeneous coordinates of a point in $P^2(\mathbb{C})^*$. ♣

[2] Cf. Example 13 from Appendix III.4.2.

2.2. Topology of coadjoint orbits.

We are mainly interested in the orbits of maximal dimension, which are called **generic** or **regular**. We already saw that every (co)adjoint orbit intersects the subspace $\mathfrak{t} \cong \mathfrak{t}^*$. Moreover, for a regular orbit Ω this intersection consists of $|W|$ different points that form a principal W-orbit. All these points are **regular** elements of $\mathfrak{t}$, i.e. their centralizer coincides with T. Therefore, all regular orbits are diffeomorphic to the full flag manifold $\mathcal{F} = K/T$.

We discuss the topology of regular orbits here.

Theorem 2 (Bruhat Lemma). *Let G be a complex semisimple Lie group, H a Cartan subgroup, and $W = N_G(H)/H$ the corresponding Weyl group. Choose a Borel subgroup $B \supset H$ and for any $w \in W$ a representative $\tilde{w} \in N_G(H)$ of the class $w \in N_G(H)/H$. Then G is the disjoint union of double B-cosets:*

$$G = \bigsqcup_{w \in W} B\tilde{w}B, \qquad \tilde{w} \in w. \tag{24}$$

□

Exercise 4.* Prove the Bruhat Lemma for the classical simple complex groups.

Hint. Use the realization of flags as filtrations and associate an element of the Weyl group to a given pair of filtrations.

In the case of $G = SL(n, \mathbb{C})$ it can be done as follows. Consider a pair of filtrations

$$f : \{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n, \quad f' : \{0\} = V_0' \subset V_1' \subset \cdots \subset V_n' = \mathbb{C}^n$$

with $\dim V_i = \dim V_i' = i$. We observe that a filtration on a vector space V induces a filtration on any subspace, on any quotient space, and on any subquotient space (i.e. a quotient of one subspace by another). In particular, the first filtration f induces a filtration on $W^{(i)} = V_i'/V_{i-1}'$. This induced filtration has the form

$$0 = W_0^{(i)} \subset W_1^{(i)} \subset \cdots \subset W_n^{(i)} = W^{(i)}$$

with

$$W_j^{(i)} = \big((V_i' \cap V_j) + V_{i-1}'\big) \Big/ V_{i-1}'.$$

Since $W^{(i)}$ is a 1-dimensional space, exactly one of the quotients $W_j^{(i)}/W_{j-1}^{(i)}$ is non-zero. Let it be for $j = s(i)$. We get a map $i \mapsto s(i)$, which in fact is

a permutation of the set $\{1, 2, \ldots, n\}$. But the set S_n of permutations is exactly the Weyl group for $SL(n, \mathbb{C})$. ♣

Let $N = \exp \mathfrak{n}$. Then N is a normal subgroup in B equal to the commutator $[B, B]$. It is usually called the **unipotent radical** of B. In fact, B is a semidirect product $H \ltimes N$.

Let $N_w = N \cap \tilde{w} N \tilde{w}^{-1}$ where $\tilde{w}$ is a representative in $N_K(T)$ of the class $w \in W = N_K(T)/T$. Define the **length** of $w \in W$ to be $l(w) = \dim_{\mathbb{C}} N - \dim_{\mathbb{C}} N_w$. According to Proposition 6 in Appendix III.3.1, this definition is equivalent to each of the following:

a) $l(w)$ is the length of the minimal representation of w as a product of canonical generators $s_1, \ldots, s_l$ of W.

b) $l(w) = \#(w(R_+) \cap R_-)$, i.e. the number of positive roots α such that $w(\alpha)$ is negative.

Corollary. *The full flag manifold $\mathcal{F} = G/B$ is a union of even-dimensional cells:*

$$\mathcal{F} = \bigsqcup_{w \in W} \mathcal{F}_w, \qquad \dim \mathcal{F}_w = 2l(w). \tag{25}$$

Proof. The decomposition (25) shows that there are $|W|$ B-orbits in $\mathcal{F}$ labelled by elements of W. Namely, put $\mathcal{F}_w = B\widetilde{w} \bmod B$. A generic element $x \in \mathcal{F}_w$ can be written as $x = nh\tilde{w} \bmod B$, $h \in H$, $n \in N$. Since $\tilde{w} \in N_G(H)$, we can rewrite it as $x = n\tilde{w} \bmod B$. Moreover, the element n in this expression is defined modulo $N_w = \tilde{w} N \tilde{w}^{-1}$. Therefore, $\mathcal{F}_w = N/N_w \cong \mathbb{C}^{l(w)} \cong \mathbb{R}^{2l(w)}$. □

In particular, since $l(w) = 1$ only for the generators, we get the following list of expressions for the second Betti number $b_2(\mathcal{F}) := \dim H^2(\mathcal{F}, \mathbb{R})$:

$$b_2(\mathcal{F}) = \text{number of simple roots} = \operatorname{rk} G = \dim_{\mathbb{C}} \mathfrak{h} = \dim T. \tag{26}$$

This important fact can also be proved as follows.

It is well known that for a compact Lie group K with $\pi_1(K) = 0$ we also have $\pi_2(K) = 0$. (It follows, for instance, from the absence of an Ad-invariant skew-symmetric bilinear form on $\mathfrak{k}$.) The exact sequence (see formula (54) in Appendix III.4.2)

$$\cdots \to \pi_k(G) \to \pi_k(\mathcal{F}) \to \pi_{k-1}(T) \to \pi_{k-1}(G) \to \cdots$$

shows that the flag manifold $\mathcal{F}$ is simply connected and

$$H_2(\mathcal{F}, \mathbb{Z}) \cong \pi_2(\mathcal{F}) \cong \pi_1(T) \cong \widehat{T} \cong \mathbb{Z}^{\dim T}. \tag{27}$$

Much more precise results can be obtained using the structure of the Weyl group (see [**B**] or Appendix III.3.1). For instance, we get the simple formula for the **Poincaré polynomial** of $\mathcal{F}$ in terms of exponents:

$$(28) \qquad P_{\mathcal{F}}(t) := \sum_k t^k \cdot \dim H^k(\mathcal{F}, \mathbb{R}) = \prod_{i=1}^{n} \frac{1 - t^{2(m_i+1)}}{1 - t^2}.$$

Now let $\Omega \subset \mathfrak{k}^*$ be a regular coadjoint orbit. The canonical symplectic form σ on Ω defines a cohomology class $[\sigma] \in H^2(\mathcal{F}, \mathbb{R})$.

Recall that an orbit $\Omega \subset \mathfrak{k}^*$ is called **integral** (see Chapter 1, Section 2.4) if $[\sigma] \in H^2(\mathcal{F}, \mathbb{Z}) \subset H^2(\mathcal{F}, \mathbb{R})$. The number of "integrality conditions" for an orbit of a given type is equal to the second Betti number of this orbit. But we have seen that this number is equal to the dimension of the manifold $\mathcal{O}_{reg}(K) \cong T_{reg}/W$ of regular orbits.

So, for compact groups the integral regular orbits form a discrete set. A more precise result is given below in Section 3.1.

We conclude this section with a résumé of the Cartan theory of highest weights.

Theorem 3 (E. Cartan). *Let G be a simply connected complex semisimple group.*

a) *Any finite-dimensional irreducible holomorphic representation (π, V) of G has a* **highest weight** *λ (i.e. λ is bigger than all other weights in the sense of the partial order introduced in Appendix* III.3.1).

b) *Two representations with the same highest weight are equivalent.*

c) *The set of possible highest weights coincides with the set P_+ of dominant weights.*

Corollary. *For a compact simply connected group K the set $\widehat{K}$ of all unirreps is labelled by the set P_+ of dominant weights.*

Proof of the theorem. a) We shall use the Bruhat Lemma from the previous section. It implies that the subset $G' \subset G$ defined as

$$G' = B_- \cdot B = N_- \cdot H \cdot N$$

is open and dense in G.

Let (π, V) be an irreducible finite-dimensional representation of the group G. We choose a maximal weight λ from among all the weights of this representation. Here "maximal" means that there is no weight of (π, V) which is bigger than λ. Let $v \in V$ be a corresponding weight vector.

Also choose a minimal weight μ from among the weights of the dual representation (π^*, V^*) and denote by $v^* \in V^*$ a corresponding weight vector.

Note that we do not suppose that the weights λ and μ are uniquely defined.

Consider the function f_{v,v^*} on G (a matrix element of π) given by

$$f_{v,v^*}(g) = \langle v^*,\, \pi(g)v\rangle.$$

Lemma 6. *The vector v is invariant under $\pi(N_+)$, while the vector v^* is invariant under $\pi(N_-)$.*

Proof. We show that $\pi_*(\mathfrak{n}_+)v = 0$ and $\pi_*(\mathfrak{n}_-)v^* = 0$. Indeed, for $\alpha \in R_+$ the operators $\pi(X_\alpha)$ increase the weight, while operators $\pi(X_{-\alpha})$ decrease the weight. □

Using Lemma 6 we can explicitly compute the function f_{v,v^*} on G' in two different ways:

$$f_{v,v^*}(n_-hn) = \begin{cases} \langle \pi^*(n_-)^{-1}v,\, \pi(h)\pi(n)v\rangle = e^{\lambda}(h)\langle v^*,\, v\rangle, \\ \langle \pi^*(h)^{-1}\pi^*(n_-)^{-1}v^*,\, \pi(n)v\rangle = e^{-\mu}(h)\langle v^*,\, v\rangle. \end{cases} \tag{29}$$

Lemma 7. *The quantity $\langle v^*,\, v\rangle$ is non-zero.*

Proof. Indeed, otherwise we have $f_{v,v^*}\big|_{G'} = 0$ and, since G' is dense in G, $f_{v,v^*} = 0$ everywhere. But $(\pi,\, V)$ is irreducible, hence the vectors of the form $\pi(g)v$, $g \in G$, span the whole space V.

Using the property $f_{\pi(g_1)v,v^*}(g) = f_{v,v^*}(gg_1)$, we conclude that $f_{v,v^*} = 0$ for all $v \in V$. This contradicts $v^* \neq 0$. □

From Lemma 7 and (29) we conclude that $\lambda = -\mu$. But we have chosen λ as any maximal weight of $(\pi,\, V)$ and μ as any minimal weight of $(\pi^*,\, V^*)$. Therefore, both weights are uniquely defined.

Moreover, since v and v^* are chosen arbitrarily in the corresponding eigenspaces, Lemma 7 and (29) imply that both λ and μ have multiplicity 1.

Now, we prove that λ is not only maximal, but the highest weight of $(\pi,\, V)$. Assume the converse. Then there exist weights λ' which do not satisfy the inequality $\lambda \geq \lambda'$. Choose a maximal weight λ_1 among them. It is clear that λ_1 is a maximal element of $Wt(\pi_\lambda)$. (Otherwise we have $\lambda_2 > \lambda_1$ for some $\lambda_2 \leq \lambda$, which is impossible.) But there is only one maximal weight, hence $\lambda_1 = \lambda$, which is also impossible.

Another proof of the statement can be obtained using the universal enveloping algebra. Namely, the space V_λ is an irreducible $U(\mathfrak{g})$-module.

From the Poincaré-Birkhoff-Witt theorem we conclude that the map $U(\mathfrak{n}_-)\otimes U(\mathfrak{h})\otimes U(\mathfrak{n}_+) \longrightarrow U(\mathfrak{g})$, given by multiplication, is an isomorphism of vector spaces. Since the highest weight vector v_λ is annihilated by elements of $\pi_*(\mathfrak{n}_+)$ (cf. proof of Lemma 6) and an eigenvector for $\pi_*(\mathfrak{h})$, we see that $V_\lambda = U(\mathfrak{n}_-)v_\lambda$. It follows immediately that $Wt(\pi_\lambda) \subset \lambda - Q_+$.

b) We make use of the following well-known fact.

Proposition 4. *An irreducible representation of G is completely determined by any of its matrix elements.*

Proof. The same arguments show that the subspace $M(\pi)$ spanned by left and right shifts of $f_{v^*,v}$ contains all other matrix elements of π. The group $G \times G$ acts on G by left and right shifts and this action gives rise to a linear representation Π of $G \times G$ in $M(\pi)$. It is easy to check that $\Pi \equiv \pi \times \pi^*$. So, $M(\pi)$ under the action of $G \times 1 \subset G \times G$ splits into irreducible subspaces where G acts by π. □

To prove c), we consider the representation (π_*, V) of the Lie algebra $\mathfrak{g}$ and restrict it to a subalgebra $\mathfrak{g}(\alpha)$, $\alpha \in R_+$. Then it splits into irreducible components with respect to $\mathfrak{g}(\alpha)$. Since the multiplicity of λ is 1, exactly one component contains the vector v.

Certainly, v remains a highest weight vector for the restriction. But for the Lie algebra $\mathfrak{g}(\alpha) \simeq \mathfrak{sl}(2, \mathbb{C})$, statement c) is true (see Appendix III.3.2). Therefore we get $\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}_+$. (Here we use the fact that the quantity in question does not depend on the choice of an Ad-invariant bilinear form on $\mathfrak{g}(\alpha)$.) Since this is true for all positive roots α, the weight λ is dominant.

Finally, let λ be a dominant weight. We can define the function f_λ on G' by

$$f_\lambda(n_- hn) = e^\lambda(h).$$

Considering the restriction of f on the subgroups $G_i = \exp \mathfrak{g}(\alpha_i)$, one can prove that this function has a continuous extension on all Bruhat cells of codimension 1. But an analytic function cannot have singularities on a complex submanifold of codimension > 1. So, f is regular everywhere. One can check that the subspace spanned by left shifts of this function is finite dimensional (in the appropriate coordinates it consists of polynomials of bounded degrees). Moreover, the corresponding representation is irreducible and has highest weight λ. □

The Cartan theorem not only classifies the finite-dimensional irreducible holomorphic representations of G and the unirreps of K but also provides an explicit construction of them.

Example 6. Let $G = SL(n+1, \mathbb{C})$, let $K = SU(n+1)$, and let H be the diagonal Cartan subgroup. A general element of H in the canonical coordinates introduced in Example 4 has the form

$$h(t) = \begin{pmatrix} t_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & t_2 t_1^{-1} & 0 & \dots & 0 & 0 \\ 0 & 0 & t_3 t_2^{-1} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & t_n t_{n-1}^{-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & t_n^{-1} \end{pmatrix}.$$

For $\lambda = (k_1, \dots, k_n) \in P \cong \mathbb{Z}^n$ we have $e^\lambda(h(t)) = t_1^{k_1} \cdots t_n^{k_n}$. Let $\Delta_k(g)$ denote the k-th principal minor of the matrix $g \in G$. Then $\Delta_k(h(t)) = t_k$. It follows that

$$f_\lambda(g) = \Delta_1^{k_1}(g) \cdots \Delta_n^{k_n}(g),$$

which is a holomorphic function on G exactly when all k_i are non-negative, i.e. when λ is dominant.

Consider the space V_λ spanned by left shifts of f_λ. It is easy to understand that this space consists of polynomials on G that are homogeneous of degree k_1 in elements of the first column, homogeneous of degree k_2 in minors of the first two columns, etc.

The group G acts on V_λ by left shifts and the corresponding representation π_λ is irreducible because it contains the unique highest weight vector f_λ of weight λ.

Note that π_λ is a subrepresentation of the left regular representation of G.

The particular case $n = 2$ is already non-trivial and still very transparent. We recommend that beginners (or non-experts) look first at this case.

Here we study the next case, $n = 3$. Let $g \in SL(3, \mathbb{C})$. Introduce the notation:

$$x_i(g) = g_{i1}, \qquad x^i(g) = \epsilon^{ijk} g_{j1} g_{k2}, \qquad 1 \le i \le 3.$$

These six polynomials on G are not independent and satisfy a unique quadratic relation

$$\sum_i x_i x^i = 0. \tag{30}$$

The space $V_{k,l}$ consists of bihomogeneous polynomials of degree k in x_i and degree l in x^j. If x_i and x^j are independent, the dimension of $V_{k,l}$ will

be equal to the product $\binom{k+2}{2} \cdot \binom{l+2}{2}$. Taking into account (30), we get the genuine dimension

$$\dim P_{k,l} = \binom{k+2}{2} \cdot \binom{l+2}{2} - \binom{k+1}{2} \cdot \binom{l+1}{2} = \frac{(k+1)(l+1)(k+l+2)}{2}.$$

$\diamondsuit$

Exercise 5. Show that this result is in agreement with the general Weyl formula (13).

Hint. Draw a picture of the root system for $SL(3, \mathbb{C})$, find fundamental weights ω_1, ω_2, and show that positive roots are

$$\alpha = 2\omega_1 - \omega_2, \qquad \beta = -\omega_1 + 2\omega_2, \qquad \gamma = \rho = \omega_1 + \omega_2.$$

So, if $\lambda = k\omega_1 + l\omega_2$, we have

$$\frac{(\lambda+\rho,\,\alpha)}{(\rho,\,\alpha)} = k+1, \quad \frac{(\lambda+\rho,\,\beta)}{(\rho,\,\beta)} = l+1, \quad \frac{(\lambda+\rho,\,\gamma)}{(\rho,\,\gamma)} = \frac{k+l+2}{2}.$$

$\clubsuit$

3. Orbits and representations

3.1. Overlook.

We keep the notation of the previous sections. In particular, K denotes a compact simply connected Lie group with a maximal torus $T \cong \mathbb{T}^n$ endowed with canonical coordinates $(t_1, \ldots, t_n)$.

We have seen that the unirreps (π_λ, V_λ) of K are labelled by dominant weights $\lambda \in P_+ \subset i\mathfrak{t}$. So, the set $\widehat{K}$ of unirreps has the form $\mathbb{Z}_+^n$, $n = \operatorname{rk} K$.

On the other hand, the set of coadjoint orbits is parametrized by points of the positive Weyl chamber $C_+ \simeq \mathbb{R}_+^n$. We shall see in a moment that integral orbits form the discrete set $\mathcal{O}_{int}(K)$, which can be identified with the set P_+ of dominant weights. The integral orbits of maximal dimension form the subset $\mathcal{O}_{int}^{reg}(K)$ labelled by the set $P_{++} = P_+ + \rho$ of regular dominant weights.

We want to relate the set $\widehat{K}$ of unirreps to some set of coadjoint orbits.

To illustrate the problem, we start with the simplest example $K = SU(2, \mathbb{C})$. We give a detailed description of all general notions for this particular case.

The general element of K has the form $k = \begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix}$, $|u|^2 + |v|^2 = 1$.

The Lie algebra $\mathfrak{k}$ has the canonical basis (X, Y, Z) such that

$$A_{x,y,z} := xX + yY + zZ = \frac{1}{2}\begin{pmatrix} iz & x+iy \\ -x+iy & -iz \end{pmatrix}.$$

The dual space $\mathfrak{k}^*$ consists of matrices $F_{a,b,c} = \begin{pmatrix} -ic & -a-ib \\ a-ib & ic \end{pmatrix}$ so that

$$\langle F_{a,b,c}, A_{x,y,z} \rangle = \operatorname{tr}(F_{a,b,c} \cdot A_{x,y,z}) = ax + by + cz.$$

The maximal torus has elements of the form $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ where t is the canonical complex coordinate with $|\,t\,| = 1$.

The lattice $\exp^{-1}(e) \cap \mathfrak{t}$ is generated by the element $\begin{pmatrix} 2\pi i & 0 \\ 0 & -2\pi i \end{pmatrix} = 4\pi Z$; the lattice $\exp^{-1}(C) \cap \mathfrak{t}$ is generated by $2\pi Z$.

The elements of $\mathfrak{g}$ and $\mathfrak{g}^*$ have the same form $A_{x,y,z}$ and $F_{a,b,c}$ as above but with complex coordinates (x, y, z) and (a, b, c), respectively. In particular, the only simple positive root α, the only fundamental weight ω, and the element ρ are

$$\alpha = F_{0,0,i}, \qquad \omega = \rho = \frac{1}{2}\alpha = F_{0,0,\frac{i}{2}}.$$

The coadjoint orbits have the form

$$\Omega_r = \left\{ F_{a,b,c} \;\middle|\; a^2 + b^2 + c^2 = r^2, \quad r \geq 0 \right\}.$$

The regular orbits are those for which $r > 0$.

The coadjoint action of $\mathfrak{k}$ on $\mathfrak{k}^*$ is given by

$$K_*(X) = b\partial_c - c\partial_b, \qquad K_*(Y) = c\partial_a - a\partial_c, \qquad K_*(Z) = a\partial_b - b\partial_a.$$

The canonical symplectic form on a coadjoint orbit is given by

$$\sigma = \frac{a\, db \wedge dc + b\, dc \wedge da + c\, da \wedge db}{a^2 + b^2 + c^2}.$$

The symplectic volume of an orbit is $vol(\Omega_r) = \int_{\Omega_r} \sigma$. Using the "geographical coordinates" $-\pi < \phi < \pi, \quad -\pi/2 < \theta < \pi/2$, we get

$$a = r\cos\theta\cos\phi, \quad b = r\cos\theta\sin\phi,$$
$$c = r\sin\theta, \quad \sigma = r\cos\theta\; d\phi \wedge d\theta.$$

Therefore, $vol(\Omega_r) = 4\pi r$. We see that this volume is an integer exactly when $r \in \frac{1}{4\pi}\mathbb{Z}$, i.e. when

$$\Omega_r \cap \mathfrak{t} = \{\pm F_{0,0,ir}\} \in \frac{1}{2\pi i}\mathbb{Z} \cdot \omega = \frac{1}{2\pi i}P.$$

Thus, the set $\mathcal{O}_{int}$ of all integral coadjoint orbits for K is naturally labelled by $P_+ \simeq \mathbb{Z}_+$, while the set $\mathcal{O}_{int}^{reg}$ of integral regular orbits is labelled by the set $P_{++} = \rho + P_+ \simeq 1 + \mathbb{Z}_+$, which is just the shift of P_+.

From Appendix III.3.2 we know that the unirrep π_n of K has $n+1$ simple weights $n\omega$, $(n-2)\omega$, ..., $-n\omega$ and that $\pi_n \otimes \pi_m = \bigoplus_{s=0}^{\min(m,n)} \pi_{m+n-2s}$.

Comparing these results with Rules 2, 3, and 5 of the User's Guide (or with formula (4) of Chapter 3), we see that the unirrep π_n must be associated with the orbit passing through the element $\frac{n\omega}{2\pi i}$. To simplify the formulations we suggest the following notation:

(N) $\Omega_\lambda \subset \mathfrak{k}^*$ *denotes the orbit passing through the point* $\frac{1}{2\pi i}\lambda$.

Remark 3. This notation needs some comments. According to the Pontrjagin duality theorem, there exists a canonical isomorphism between any abelian locally compact group A and its second dual $\widehat{\widehat{A}}$.

But often (e.g., for all finite groups and for additive groups of locally compact fields) already the first dual $\widehat{A}$ is isomorphic to A. However, in this case there is no canonical isomorphism.

In particular, there is no canonical way to label the characters of $\mathbb{R}$ by elements of $\mathbb{R}$. Practically, two special ways are used:

$$1)\ \chi_\lambda(x) = e^{i\lambda x} \qquad \text{and} \qquad 2)\ \chi_\lambda(x) = e^{2\pi i\lambda x}.$$

Both ways have their advantages and disadvantages. In this book we use the second way. It is worthwhile to discuss what happens if we choose the first one.

The 1-dimensional representations $U_{F,H}$, which are used in Rule 2 to construct the unirreps, would acquire the simpler form

$$U_{F,H}(\exp X) = e^{i\langle F,\, X\rangle}.$$

The notation (N) above would also change to a more natural one:

(N′) $\Omega_\lambda \subset \mathfrak{k}^*$ *denotes the orbit passing through the point* $-i\lambda$.

But, on the other hand, the symplectic structure on coadjoint orbits would be related not to the form B_F but to the form $\frac{1}{2\pi}B_F$. The desire to keep the initial definition of the symplectic form σ, which is now widely known as the **canonical** or **Kirillov-Kostant form**, forces us to introduce the rather clumsy definition (N). ♡

In the case of $K = SU(2, \mathbb{C})$ the notation (N) leads to the correspondence $\pi_n \longleftrightarrow \Omega_{n\omega}$, which is naturally generalized to the correspondence $\pi_\lambda \longleftrightarrow \Omega_\lambda$ for general compact semisimple groups.

On the other hand, let us try to use Rule 6 to compute the generalized character of π_n at the point $k = \exp A_{0,0,z} = \exp zZ$.

The key ingredient is the integral

$$\int_{\Omega_{r\omega}} e^{2\pi i\langle F, X\rangle}\sigma = \frac{r}{4\pi}\int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} e^{\frac{1}{2}irz\sin\theta}\cos\theta d\theta = \frac{2}{z}\sin\frac{rz}{2}.$$

Therefore, taking into account the equality

$$p(\exp zZ) = \frac{\sinh\bigl(\mathrm{ad}\,(zZ/2)\bigr)}{\mathrm{ad}\,(zZ/2)} = \frac{\sin z/2}{z/2},$$

we get the final formula

$$\frac{1}{p(zZ)}\int_{\Omega_{r\omega}} e^{2\pi i\langle F, zZ\rangle}\sigma = \frac{\sin rz/2}{\sin z/2}.$$

But we know that

$$\chi_{\pi_n}(\exp zZ) = \frac{\sin(n+1)z/2}{\sin z/2}.$$

This suggests we assign the unirrep π_n to the orbit $\Omega_{(n+1)\omega}$. This clearly generalizes to the correspondence $\pi_\lambda \longleftrightarrow \Omega_{\lambda+\rho}$ for general compact semisimple groups.

Another argument in favor of this correspondence comes from the consideration of infinitesimal characters. We have seen in Section 1.1 that the infinitesimal character of π_n is given by $I_n(\Delta) = \frac{n(n+2)}{8} = \frac{(n+1)^2}{8} - \frac{1}{8}$. More generally (see Theorem 1e)), for any compact semisimple group K we have $I_\lambda(\Delta) = |\lambda+\rho|^2 - |\rho|^2$, an expression which explicitly contains $\lambda + \rho$.

For other elements of $Z(\mathfrak{g})$ their infinitesimal characters are also better expressed in terms of the orbit $\Omega_{\lambda+\rho}$, rather than Ω_λ (see Section 3.6).

3.2. Weights of a unirrep.

Let (π_λ, V_λ) be the unirrep of K with the highest weight $\lambda \in P_+$. The restriction $\mathrm{Res}_T^K \pi_\lambda$ is no longer irreducible. Since T is abelian, it splits into 1-dimensional unirreps. If we introduce in T the canonical coordinates (10) and identify P with $\mathbb{Z}^n$ using the basis of fundamental weights, then the 1-dimensional unirreps in question are exactly the functions e^μ, $\mu \in P$, defined by (11). For $\mu = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ we have

$$e^\mu(t) = t_1^{m_1}\cdots t_l^{m_l} \qquad (t^\mu \text{ for short}).$$

Denote by V_λ^μ the isotypic component of type μ in V_λ:

$$V_\lambda^\mu := \{v \in V_\lambda \mid \pi_\lambda(t)v = t^\mu v\}.$$

The weights of a given representation (π, V) form a **multiset** (i.e. a set with the additional structure: to every element of the set we associate a positive integer, the **multiplicity** of this element). We denote this multiset by $Wt(\pi)$.

One of the oldest (and most needed for applications) problems in the representation theory of compact Lie groups is to compute the **multiplicity** $m_\lambda(\mu)$ **of the weight** $\mu \in Wt(\pi_\lambda)$, i.e. the dimension of V_λ^μ.

In terms of the restriction functor we can write:

$$\mathrm{Res}_T^K \pi_\lambda = \sum_{\mu \in P} m_\lambda(\mu) e^\mu.$$

There are several different formulae for this quantity, but no one of them is efficient enough. The orbit method, unfortunately, is no exception. The formula suggested by this method is elegant and transparent but (at least in its present form) not practical.

Example 7. To show the flavor of this problem, we consider the case $G = SU(3)$. Here the weight lattice is the usual triangular lattice in $\mathbb{R}^2$ generated by two fundamental weights ω_1 and ω_2. The multiplicity function $m_\lambda(\mu)$, for two specific cases $\lambda = 4\omega_1 + 2\omega_2$ and $\lambda = 3\omega_1 + 3\omega_2$, looks like:

$$\begin{array}{ccccccccccccc}
 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & 2 & & 2 & & 1 & \\
1 & & 2 & & 3 & & 3 & & 3 & & 2 & & 1 \\
 & 1 & & 2 & & 3 & & 3 & & 2 & & 1 & \\
 & & 1 & & 2 & & 3 & & 2 & & 1 & & \\
 & & & 1 & & 2 & & 2 & & 1 & & & \\
 & & & & 1 & & 1 & & 1 & & & &
\end{array}
\qquad
\begin{array}{ccccccccccccc}
 & & & 1 & & 1 & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 2 & & 2 & & 1 & & \\
 & 1 & & 2 & & 3 & & 3 & & 2 & & 1 & \\
1 & & 2 & & 3 & & 4 & & 3 & & 2 & & 1 \\
 & 1 & & 2 & & 3 & & 3 & & 2 & & 1 & \\
 & & 1 & & 2 & & 2 & & 2 & & 1 & & \\
 & & & 1 & & 1 & & 1 & & 1 & & &
\end{array}$$

One can easily guess from these pictures the general principle which is valid for any unirrep of $SU(3)$.

Namely, let $Q \subset P$ be the root sublattice. For $G = SU(3)$ it has index 3 in P. The support of $m_\lambda(\mu)$, i.e. the set of $\mu \in P$ for which $m_\lambda(\mu) \neq 0$, is a convex hexagon in $\lambda + Q$, whose vertices form a W-orbit of the highest weight λ. The values of $m_\lambda(\mu)$ along the perimeter of the hexagon are equal to 1. On the next layer they are equal to 2 and continue to grow linearly until the hexagonal layer degenerates to an isosceles triangle. Then the growth stops (on the value $\min(k,\ l) + 1$ for $\lambda = k\omega_1 + l\omega_2$).

We recommend that the reader compare this simple description with the computations based on the Weyl or the Kostant formula. ♢

For general compact groups the situation is described by the following theorem.

Theorem 4. *The support of m_λ has the form*

$$\operatorname{supp} m_\lambda = \bigcap_{w\in W} w(\lambda - Q_+). \tag{31}$$

Sketch of the proof. We already mentioned that all weights of π_λ are contained in the set $\lambda - Q_+$. Since supp m_λ is invariant under the action of the Weyl group, we conclude that supp $m_\lambda \subset \bigcap_{w\in W} w(\lambda - Q_+)$.

The inverse inclusion can be proved by considering the restriction of π_λ to different 3-dimensional subalgebras $\mathfrak{g}(\alpha)$. □

Remark 4. The convexity of supp m_λ follows from the remarkable recent result by Okounkov [**Ok**] who showed that the function $\log m_\lambda$ is concave. ♡

The ideology of the orbit method relates the restriction functor Res_T^K with the natural projection $p:\ \mathfrak{k}^* \to \mathfrak{t}^*$. The reader can see the relation comparing Theorem 4 with the following well-known fact.

Theorem 5. *The image of Ω_λ under the projection $p:\ \mathfrak{g}^* \longrightarrow \mathfrak{t}^*$ is the convex hull of the intersection $\Omega_\lambda \cap \mathfrak{t}^*$, which consists of points $\frac{1}{2\pi i} w(\lambda)$, $w \in W$.*

Sketch of the proof. First, we mention that the statement of the theorem follows from a more general result about Hamiltonian actions of a torus on a symplectic manifold (see [**A**] and [**GS2**]). Here we briefly describe the other approach.

We identify $\mathfrak{k}^*$ with $\mathfrak{k}$ as in Section 2.1 and write the general point $X \in \mathfrak{k}$ in the form

$$X = X_0 + \sum_{\alpha\in R_+} (c_\alpha X_\alpha - \bar{c}_\alpha X_{-\alpha}), \tag{32}$$

for $X_0 \in \mathfrak{t}$ and $c_\alpha \in \mathbb{C}$.

Let us study the singular locus of the projection map $p:\ \Omega \longrightarrow \mathfrak{t}:\ X \mapsto X_0$. It is the image by p of those points $X \in \Omega$ for which the tangent map $p_*:\ T_X\Omega \longrightarrow \mathfrak{t}$ is not surjective. Geometrically, it is the set of points X_0 where the preimage $p^{-1}(X_0)$ may differ from the preimage of nearby points. In particular, the boundary of $p(\Omega)$ is contained in the singular locus.

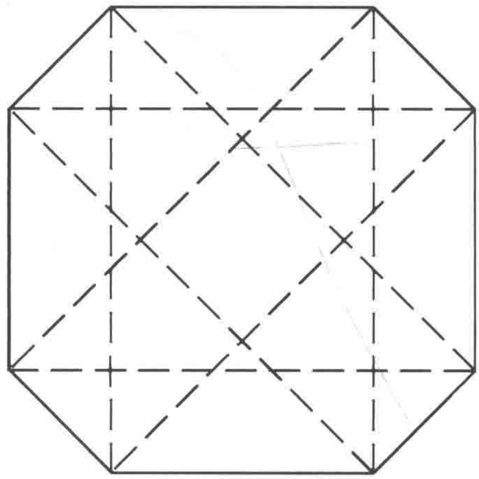

Figure 5

The tangent space $T_X\Omega$ consists of elements $[Y, X]$, $Y \in \mathfrak{k}$. If the image $p_*(T_X\Omega)$ is not the whole space $\mathfrak{t}$, then $\big(p_*(T_X\Omega), Z\big)_K = 0$ for some non-zero $Z \in \mathfrak{t}$. This means that for all $Y \in \mathfrak{k}$ we have $([Y, X], Z)_K = 0$, which implies $(Y, [X, Z])_K = 0$, hence $[X, Z] = 0$. Since

$$[Z, X] = \sum_{\alpha \in R_+} \alpha(Z)(c_\alpha X_\alpha + \overline{c}_\alpha X_{-\alpha}),$$

the coefficients $c_{\pm\alpha}$ must vanish for all $\alpha \in R_+$ with $\alpha(Z) \neq 0$.

In Section 2.1 we introduced the family of root systems $R^{(m)}$, $1 \leq m \leq M$, which are all possible intersections of R with hyperplanes in $i\mathfrak{t}$. We keep the notation of that section here.

We say that a singular point $p(X)$ is of type m if $X \in \mathfrak{k}^{(m)}$. Note that any such point is conjugate by an element of $K^{(m)}$ to a point in $\mathfrak{t} \cap \Omega$, i.e. to one of the points of the set $S = \{\frac{1}{2\pi i} w(\lambda) \mid w \in W\}$. So, the set of points of type m is a union of projections of finitely many $K^{(m)}$-orbits in $\mathfrak{k}$. By induction, we can assume that these projections are convex hulls of $W^{(m)}$-orbits in S where $W^{(m)}$ is the Weyl group of $\mathfrak{g}_1^{(m)}$.

We see that the set of all singular points forms finitely many convex polytopes of dimension $< n$ with vertices in S. In particular, the boundary of $p(\Omega_\lambda)$ consists of polytopes of codimension 1 that correspond to root systems of rank $n-1$ in R.

From this one can deduce the statement of the theorem. □

Example 8. For a group of type $\mathbf{B_2} \cong \mathbf{C_2}$ the projection of a typical orbit and the set of singular points is shown in Figure 5. ♢

Comparing Theorems 4 and 5, we get

Theorem 6. *Let K be a simply connected compact Lie group, and let T be its maximal torus. Then Rule 3 of the User's Guide holds for the pair $T \subset K$ if we associate the orbit Ω_λ to a unirrep π_λ and consider in $p(\Omega_\lambda)$ only points which are congruent to λ mod Q.* □

Recall that we use the notation (N) introduced on page 163.

3.3. Functors Ind and Res.

Now we consider the decomposition of a tensor product of two unirreps. This is also an old problem that is important in many applications. Recall that Rule 5 relates the tensor product of unirreps π_{Ω_1} and π_{Ω_2} with the arithmetic sum $\Omega_1 + \Omega_2$. The most famous and visual case $K = SU(2)$ was discussed above. Here we describe its physical and geometric interpretations.

Example 9. The so-called **addition rule** for moments or spins in quantum mechanics in mathematical language means that

$$\pi_m \otimes \pi_n = \bigoplus_{|m-n|}^{m+n} \pi_k, \quad k \equiv (m+n) \operatorname{mod} 2.$$

This is in perfect agreement with the following geometric fact in $\mathbb{R}^3$: the arithmetic sum of a sphere of radius r_1 with a sphere of radius r_2 is the union of spheres of radius r where $|r_1 - r_2| \leq r \leq r_1 + r_2$. ♢

As was shown recently in [**Kl**] (see also [**KT**] and [**AW**]), Rule 5 with the same amendment holds for all compact Lie groups of type **A**. It can be formulated as follows.

Proposition 5. *Let Ω_1, Ω_2, Ω_3 be any three integral orbits in $\mathfrak{su}(n)$, and let π_1, π_2, π_3 be the corresponding unirreps of $SU(n)$. Then $\Omega_1 + \Omega_2$ contains Ω_3 iff $\pi_1 \otimes \pi_2$ contains π_3.* □

All this is strong evidence in favor of the first correspondence: $\pi_\lambda \longleftrightarrow \Omega_\lambda$. However there are some facts that have a better explanation in terms of the second correspondence.

Let us return to the multiplicity problem. It is known that $m_\lambda(\,\cdot\,)$ is a piecewise polynomial function, as in Example 7. This follows, e.g., from the Kostant formula (15). Moreover, the domains where the multiplicity is given by a fixed polynomial are known. They are bounded by hyperplanes

$$P_{w,m} = w(\lambda) + \mathfrak{t} \cap \mathfrak{k}_1^{(m)}, \ w \in W, \quad \operatorname{rk} R^{(m)} = n - 1. \tag{33}$$

On the other hand, by Theorem 5 the image of Ω_λ under the canonical projection $p : \mathfrak{g}^* \to \mathfrak{t}^*$ is the convex hull of the set $\{\frac{1}{2\pi i} w(\lambda),\, w \in W\}$.

Moreover, in the proof of this theorem we saw that the set of singular points of Ω_λ is the union of polytopes which are exactly the intersections of $p(\Omega_\lambda)$ with the hyperplanes (33).

In other words, the multiplicity formula changes its form exactly where the structure of the preimage $p^{-1}(X_0)$ changes.

This suggests that the weight multiplicity $m_\lambda(\mu)$ is related to the geometry of the preimage $p^{-1}(i\mu) \cap \Omega_\lambda$.

This is indeed so, but only if we replace Ω_λ by $\Omega_{\lambda+\rho}$!

Namely, let $X^\mu_{\lambda+\rho}$ be the quotient of $p^{-1}(i\mu) \cap \Omega_{\lambda+\rho}$ by the action of $Stab_K(\mu)$. This is the so-called **reduced symplectic manifold** (see [**AG**] and/or Appendix II.3.4).

It turns out that the most naive conjecture

$$m_\lambda(\mu) = vol\, X^\mu_{\lambda+\rho} \tag{34}$$

is "asymptotically true" (see [**Hec**]).

There is a vast literature surrounding the more sophisticated forms of this conjecture involving the Todd genus and the Riemann-Roch number of the manifold $X^\mu_{\lambda+\rho}$. We refer to [**GS2**], [**GLS**], and to papers quoted there.

My personal impression is that the right formula for multiplicities is still to be found. I want to make the following observation here. Let r_λ denote the density of the projection $p_*(vol)$ of the canonical measure on the orbit $\Omega_{\rho+\lambda}$. Formula (12) for the character together with the Weyl integral formula (17) imply that this density has the form

$$r_\lambda(x) = \sum_\mu m_\lambda(\mu) \cdot r_\rho(x - \mu). \tag{35}$$

We illustrate it on the simplest

Example 10. Let $K = SU(2)$. Then the lattice P can be identified with $\frac{1}{2}\mathbb{Z} \subset \mathbb{R}$. It turns out that r_λ is simply the characteristic function of the segment $[-\lambda, \lambda]$. (This property of the area of a 2-dimensional sphere was known already to Archimedes.)

In particular, $r_\rho = \chi_{[-\frac{1}{2}, \frac{1}{2}]}$. Equality (35) above takes the form:

$$\chi_{[-\frac{n+1}{2}, \frac{n+1}{2}]} \quad = \quad \sum_{k=0}^{n} \quad \chi_{[k-\frac{n}{2}-\frac{1}{2}, k-\frac{n}{2}+\frac{1}{2}]}.$$

(See Figure 6.) ◇

We recommend that the reader consider in detail the next case $K = SU(3)$. Here the projection of Ω_ρ is a regular hexagon in $\mathfrak{t}$ and the graph of

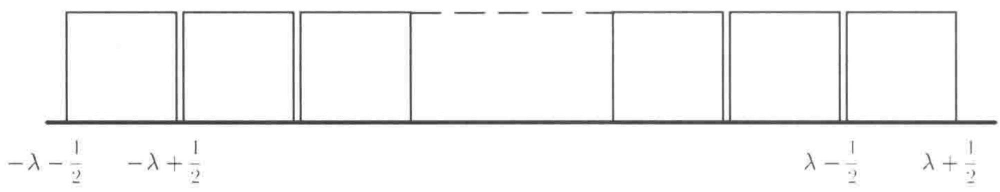

Figure 6. The projection of the canonical measure on S^2.

r_ρ is a right pyramid of volume 1 over this hexagon. Equality (35) in this case can be easily interpreted as the rule formulated in Example 7.

3.4.* Borel-Weil-Bott theorem.

This is a culminating result in the representation theory of compact Lie groups. It gives a uniform geometric construction for all unirreps of all compact connected Lie groups K. At the same time this result requires more knowledge of algebraic geometry and homological algebra than other parts of the book. I do not suggest that the unprepared reader simply skip this material, but recommend that it be considered as an excursion in a close but still foreign country.

In fact, the first part of the theory, the so-called Borel-Weil theorem, is just the modern interpretation of E. Cartan's theory of the highest weight (see Section 2.2).

This theory admits a beautiful generalization due to R. Bott. We want to show how these results agree with the ideology of the orbit method.

The Borel-Weil theory deals with homogeneous holomorphic line bundles L on flag manifolds $\mathcal{F}$, i.e. bundles which admit a K-action by holomorphic transformations of the total space and the base. This action automatically extends to an action of the complex group G. This allows us to describe everything in group-theoretic terms.

Namely, let χ be a holomorphic character (i.e. complex analytic 1-dimensional representation) of the Borel subgroup B. Let B act on G by left shifts and denote by $\mathbb{C}_\chi$ the 1-dimensional complex space where B acts via the character χ. Then the **fibered product** $G \times_B \mathbb{C}_\chi$ is a complex manifold that has a natural projection on $\mathcal{F} = G/B$ with a fiber $\mathbb{C}$ over each point. Hence, we can identify it with the total space of a line bundle L_χ over $\mathcal{F}$.

A holomorphic section $s : \mathcal{F} \to L_\chi$ is represented by a holomorphic function ϕ_s on G satisfying

$$\phi_s(bg) = \chi(b) \cdot \phi_s(g).$$

We denote by $\Gamma_{hol}(L_\chi)$ the (finite-dimensional) space of all holomorphic sections of L_χ. The group G acts on this space by right[3] shifts:

$$\phi_{g\cdot s}(g') = \phi_s(g'g).$$

The representation thus obtained is called **holomorphically induced** from $(B,\ \chi)$ and is denoted by $\operatorname{Ind} hol_B^G\ \chi$ (cf. Appendix V and Chapter 4).

It is easy to describe the set of homogeneous holomorphic bundles on $\mathcal{F}$ or, equivalently, the set of all holomorphic characters of B. Namely, these characters are trivial on $[B,\ B] = N$ and their restrictions on the Cartan subgroup H are exactly the functions e^λ, $\lambda \in P$, introduced by formula (9) in Section 1.1.

Thus, to any $\lambda \in P$ there corresponds a character of the Borel subgroup B, hence a homogeneous holomorphic line bundle on $\mathcal{F}$ which we denote by L_λ.

Theorem 7 (Borel-Weil). *The space $\Gamma_{hol}(L_\lambda)$ is non-zero exactly when $\lambda \in P_+$, the set of dominant weights, and in this case G acts on $\Gamma_{hol}(L_\lambda)$ via an irreducible representation π_λ with highest weight λ.* $\square$

This theorem strongly suggests that the representation π_λ should be related to the integral coadjoint orbit Ω_λ passing through $\frac{\lambda}{2\pi i}$. The point is that on any orbit Ω_λ there is a canonically defined line bundle $\widetilde{L}_\lambda$ associated with the symplectic form σ on the orbit. The Chern class of $\widetilde{L}_\lambda$ is exactly the cohomology class defined by σ. A more precise statement relates σ with the curvature of a connection in $\widetilde{L}_\lambda$.

Recall that in the previous section we defined the G-covariant isomorphism $\varphi : \mathcal{F} \simeq G/B \to \Omega_\lambda$. It turns out that $L_\lambda = \varphi^!(\widetilde{L}_\lambda)$, the induced bundle.

Example 11. a) The trivial representation p_0 is realized in the space of constant functions on $\mathcal{F}$, which is clearly induced by the map $p_0 : \mathcal{F} \to \{pt\} = \Omega_0$.

So, π_0 should correspond to the one-point orbit, the origin in $\mathfrak{k}$.

b) The representation $\pi_{k\omega_1}$ of $SL(3,\ \mathbb{C})$ was realized in Example 6 in the space of homogeneous polynomials of degree k in $(x^1,\ x^2,\ x^3)$. According to the Borel-Weil theorem, this space can be realized as a space of sections of a line bundle $L_{k\omega_1}$ over $\mathcal{F}$. But from the construction in Example 1 one can see that $L_{k\omega_1}$ is induced from a canonical line bundle $\widetilde{L}_{k\omega_1}$ over $\Omega_{k\omega_1} \cong \mathbb{P}^2(\mathbb{C})$. So, it is natural to associate this representation with $\Omega_{k\omega_1}$. $\diamondsuit$

Now, consider the arguments in favor of another correspondence.

[3]However, note that this is a *left action*: $s \mapsto g \cdot s$.

One of these arguments is the complement by Bott to the Borel-Weil theorem. It deals with the space $H^k(\mathcal{F},\ \mathcal{L}_\lambda)$ of k-dimensional cohomology of $\mathcal{F}$ with coefficients in the sheaf $\mathcal{L}_\lambda$ related to the line bundle L_λ.

For $k=0$ this cohomology space reduces to the space $\Gamma_{hol}(L_\lambda)$ of holomorphic sections which appears in the Borel-Weil theorem. The ideology of homological algebra suggests that we also consider the higher-dimensional cohomology each time when the zero-dimensional cohomology appears. The result of this consideration is

Theorem 8 (Bott). *The space $H^k(\mathcal{F},\ \mathcal{L}_\lambda)$ is non-zero exactly when $\lambda+\rho = w(\mu+\rho)$ for some $\mu \in P_+,\ w \in W,$ and $k = l(w)$, the length of w.*

In this case the representation of G in $H^k(\mathcal{F},\ \mathcal{L}_\lambda)$ is equivalent to π_μ.□

Example 12. a) The trivial representation π_0 occurs in the space $H^r(\mathcal{F},\ \mathcal{L}_{-2\rho})$ where $r = \#R_+$ is the number of positive roots. Indeed, here r is the complex dimension of $\mathcal{F}$ and $L_{-2\rho}$ is the line bundle of the **anticanonical class** since -2ρ equals the sum of negative roots. According to Serre duality, the r-dimensional cohomology with coefficients in $\mathcal{L}_{-2\rho}$ is isomorphic to the 0-dimensional cohomology with coefficients in the trivial sheaf $\mathcal{L}_0$.

b) The fundamental representation π_{ω_1} of $SL(3,\ \mathbb{C})$ occurs in the following cohomology spaces:

$$\begin{array}{ccc} H^0(\mathcal{F},\ \mathcal{L}_{\omega_1}), & H^1(\mathcal{F},\ \mathcal{L}_{-3\omega_1+2\omega_2}), & H^1(\mathcal{F},\ \mathcal{L}_{2\omega_1-2\omega_2}), \\ H^2(\mathcal{F},\ \mathcal{L}_{-4\omega_2}), & H^2(\mathcal{F},\ \mathcal{L}_{-4\omega_1+\omega_2}), & H^3(\mathcal{F},\ \mathcal{L}_{-2\omega_1-3\omega_2}). \end{array}$$

The first realization is just the standard one (see Examples 1 and 6). Another realization that has a transparent geometric interpretation is $H^2(\mathcal{F},\ \mathcal{L}_{-4\omega_2})$. Since $4\omega_2$ is a singular weight, the line bundle $L_{-4\omega_2}$ is induced from a bundle over $\mathbb{P}^2(\mathbb{C})$. According to Serre duality, the corresponding cohomology coincides with $H^0(\mathbb{P}^2(\mathbb{C}),\ \mathcal{L}_{\omega_1})$. ♢

The Bott theorem says that the unirrep π_λ is related not to one, but to the whole family of orbits $\{\Omega_{\lambda+\rho-w(\rho)},\ w \in W\}$. Note that these orbits are considered to be complex manifolds endowed with a Kähler structure.

On the other hand, any orbit admits not one but several Kähler structures. For regular orbits they are labelled by the elements of the Weyl group (see Remark 2 above).

We see that the correspondence between orbits and representations for compact groups is much more sophisticated than for solvable groups.

3.5. The integral formula for characters.

Another and more visual argument in favor of the correspondence $\pi_\lambda \leftrightarrow \Omega_{\lambda+\rho}$ is the equality

$$vol(\Omega_{\lambda+\rho}) = \dim \pi_\lambda, \tag{36}$$

which is in perfect agreement with the principle of quantization: the dimension of the quantum phase space is equal to the volume of the classical phase space in Planck units. In short – one dimension per volume unit.[4]

Equality (36) follows from the integral formula for the characters (the modified Rule 6) which we discuss here in the context of compact groups.

In this case the unirreps are finite dimensional and the characters are regular, even analytic, functions on the group. On the other hand, the factor $\frac{1}{p(X)}$, entering in the modified Rule 6, is defined only in the open subset $\mathcal{E} \subset \mathfrak{g}$ where the exponential map is a bijection.

Theorem 9 (see [**K11**]). *For $X \in \mathcal{E}$ we have*

$$\mathrm{tr}\ \pi_\lambda(\exp X) = \frac{1}{p(X)} \int_{\Omega_{\lambda+\rho}} e^{2\pi i \langle F, X\rangle + \sigma}. \tag{37}$$

Equality (36) is just a particular case of this theorem when $X = 0$.

Proof. All known proofs of (37) consist of two parts: 1) show that both sides are proportional; 2) check that the proportionality factor equals 1.

The first statement follows e.g. from the differential equations for generalized characters (see the next section). This is the way the integral was first computed by Harish-Chandra (Theorem 2 in [**H**], vol. II, pp. 243–276).

Another possibility uses the properties of the Fourier transform on a semisimple Lie algebra (see [**Ver2**]).

The second statement basically is equivalent to the comparison of the two volume forms on the group K: the one-form dg corresponds to the Riemannian metric on K induced by the Killing form, the other form vol is normalized by the condition $vol(K) = 1$.

It also has different proofs. One, used in [**Ki11**], is based on computing the asymptotics of the integral

$$\int_{\mathfrak{g}} e^{-t(X,X)} d\mu(\exp X)$$

[4] Here again the two identifications of $\widehat{\mathbb{R}}$ with $\mathbb{R}$ are related with two choices of the Planck constants: the usual h and the **normalized** $\hbar = \frac{h}{2\pi}$. (See Remark 3.)

when $t \to +\infty$. Another uses equality (36), which can be proved by methods of algebraic topology assuming the Borel-Weil theorem. □

Theorem 9, for the simplest case $G = SU(2)$, was already stated in [**K1**]. But it was observed in [**KV**] and [**DW**] only recently that it implies the following remarkable property of the convolution algebra on G.

Let $C^\infty(\mathfrak{g})'$ (resp. $C^\infty(G)'$) denote the space of distributions with compact support on a Lie algebra $\mathfrak{g}$ (resp. on a Lie group G). Define the transform $\Phi : C^\infty(\mathfrak{g})' \to C^\infty(G)'$ by

$$\langle \Phi(\nu),\ f\rangle \ := \ \langle \nu,\ p\cdot(f\circ \exp)\rangle. \tag{38}$$

Theorem 10 (see [**DW**]). *For* $\mathrm{Ad}(G)$*-invariant distributions the convolution operations on* G *and* $\mathfrak{g}$ *(the latter being considered as an abelian Lie group) are related by the transform above:*

$$\Phi(\mu) *_G \Phi(\nu) = \Phi(\mu *_{\mathfrak{g}} \nu). \tag{39}$$

□

So, the transform Φ "straightens" the group convolution, turning it into the abelian convolution on $\mathfrak{g}$. This implies in particular the following remarkable geometric fact.

Corollary. *For any two (co)adjoint orbits* $O_1,\ O_2 \subset \mathfrak{g}$ *the corresponding conjugacy classes* $C_1 = \exp O_1,\ C_2 = \exp O_2$ *possess the property*

$$C_1 \cdot C_2 \subset \exp(O_1 + O_2). \tag{40}$$

The analytic explanation of this geometric phenomenon can be given using the property of Laplace operators which we discuss below (see the next section).

3.6. Infinitesimal characters.

Let $Z(\mathfrak{k})$ be the center of the enveloping algebra $U(\mathfrak{k})$. An element $A \in Z(\mathfrak{k})$ can be interpreted as a distribution on K supported at $\{e\}$ or as a differential operator D_A on K acting by the formula

$$D_A f = A * f = f * A. \tag{41}$$

Therefore, for any unirrep π of K we have the equality

$$\pi(D_A f) = \pi(A * f) = \pi(A)\pi(f) = I_\pi(A)\pi(f). \tag{42}$$

It follows that all matrix elements of π and its character are eigenfunctions for D_A with the eigenvalue $I_\pi(A)$.

The integral formula (37) and equation (42) for the characters together imply the modified Rule 7. Conversely, the modified Rule 7 can be used to derive the integral formula (37).

We use these facts to get an explicit formula for the isomorphism **sym** : $Y(\mathfrak{g}) \to Z(\mathfrak{g})$. For illustration we give

Example 13. Let $G = SU(2)$ and $\mathfrak{g} = \mathfrak{su}(2)$ with the standard basis X, Y, Z obeying the standard commutation relations

$$[X,\, Y] = Z, \quad [Y,\, Z] = X, \quad [Z,\, X] = Y.$$

We denote by small letters x, y, z the same elements X, Y, Z considered as coordinates on $\mathfrak{g}^*$, and by α, β, γ the dual coordinates on $\mathfrak{g}$. Also put

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \rho = \sqrt{\alpha^2 + \beta^2 + \gamma^2}.$$

It is clear that in our case $Y(\mathfrak{g}) = \mathbb{C}\,[r^2]$ and $Z(\mathfrak{g}) = \mathbb{C}\,[C]$ where

$$C := X^2 + Y^2 + Z^2 = \mathbf{sym}(r^2) \in Z(\mathfrak{g}).$$

Unfortunately, the map **sym** : $\mathbb{C}[x,y,z] \to U(\mathfrak{g})$ is not easily computable even when restricted to $Y(\mathfrak{g}) = \mathbb{C}\ [r^2]$. E.g. one can check that $\mathbf{sym}\,(r^4) = C^2 + \frac{1}{3}C$, but the direct computation of $\mathbf{sym}\,(r^6) = C^3 + C^2 + \frac{1}{3}C$ is already rather complicated.[5] So we choose the roundabout way based on the modified Rule 7.

It is instructive to compare this problem with the computation of the various symbols of differential operators (cf. **[Ki2]**, §18, no. 2).

The function p on $\mathfrak{g}$ in our case takes the form

$$\begin{aligned} p(\alpha,\, \beta,\, \gamma) &= \left(\det\left(\frac{\sinh(\operatorname{ad}(\alpha X + \beta Y + \gamma Z)/2)}{\operatorname{ad}(\alpha X + \beta Y + \gamma Z)/2}\right)\right)^{\frac{1}{2}} \\ &= \frac{\sin(\rho/2)}{\rho/2} = 1 - \frac{\rho^2}{24} + \cdots . \end{aligned} \tag{43}$$

Recall that we identify polynomials on $\mathfrak{g}$ with differential operators on $\mathfrak{g}^*$ with constant coefficients, so that ρ^2 goes to the operator $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ and the function $j(\rho) = \frac{\sin(\rho/2)}{\rho/2}$ corresponds to some differential operator J of infinite order.

[5]In the lectures by Roger Godement on Lie groups this computation is accompanied by the remark: "Resultat qui n'incite pas à pousser plus loin les investigations, encore que les physiciens procedent tout les jours depuis le debut des années 30 à des calculs de ce genre."

Since the restriction of the operator Δ to $Y(\mathfrak{g})$ is given by the simple expression $\Delta = r^{-1} \circ \frac{d^2}{dr^2} \circ r$, we get

$$(JF)(r) = r^{-1} j\left(\frac{d}{dr}\right)(rF(r)) = F(r) - \frac{\left(rF(r)\right)''}{24r} + \dots$$

and, in particular,

$$\text{(44)} \qquad Jr^2 = r^2 - \frac{1}{4}; \qquad J\,\frac{\sinh ar}{ar} = \frac{\sin(a/2)}{a/2} \cdot \frac{\sinh ar}{ar}.$$

The modified Rule 7 implies that the map $\mathbf{sym} \circ J$ restricted on $Y(\mathfrak{g})$ is an algebra homomorphism, hence for any power series f we have

$$\text{(45)} \qquad \mathbf{sym}((Jf)(r^2)) = f\left(\mathbf{sym}(Jr^2)\right) = f\left(C - \frac{1}{4}\right).$$

From (44) and (45) we obtain

$$\text{(46)} \qquad \mathbf{sym}\left(\frac{\sinh ar}{ar}\right) = \frac{a/2}{\sin(a/2)} \cdot \frac{\sin\left(a\sqrt{\frac{1}{4} - C}\right)}{a\sqrt{\frac{1}{4} - C}}.$$

This gives the following explicit expression for the map $\mathbf{sym} : Y(\mathfrak{g}) \to Z(\mathfrak{g})$:

$$\text{(47)} \qquad \mathbf{sym}(r^{2n}) = \frac{(-1)^{n-1}}{4^n} \sum_{k=0}^{n} \binom{2n+1}{2k} B_{2k}(4^k - 2)(1 - 4C)^{n-k}$$

where B_{2k} are the Bernoulli numbers. This explicit formula shows, in particular, that there is no simple expression for the map $\mathbf{sym}$ in general. ◇

4. Intertwining operators

Another question important in applications is the structure of **intertwining**, or G-covariant, operators between two representation spaces. Many remarkable differential and integral operators can be interpreted (or even defined) as intertwining operators. For example, the Laplace operator

$$\Delta = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^2$$

is an intertwining operator for the natural action of the Euclidean motion group in any function space on $\mathbb{R}^n$. Other examples are the Fourier and Radon transforms.

The big deficiency of the orbit method is that, up to this point, it has helped very little in the study of intertwining operators. (Although there is a beautiful formula for the so-called **intertwining number** of two representations in terms of symplectic geometry – see [**GS1**], [**GS2**], [**GLS**].)

The reason is that the natural correspondence exists between orbits and equivalence classes of unirreps rather than unirreps themselves. The construction of an individual representation π from a given equivalence class defined by an orbit Ω needs some arbitrary choice (e.g., the choice of a representative $F \in \Omega$ and a subalgebra $\mathfrak{h}$ subordinate to F).

Recently a new approach to the representation-theoretic study of special functions was suggested in [**EK**] and further developed in [**EFK**]. It requires a detailed description of intertwining operators for certain geometric representations of compact groups. I consider the application of the orbit method to these questions to be a very challenging problem.

Chapter 6

Miscellaneous

In this chapter we collect information about how the orbit method works in situations different from solvable and compact Lie groups.

There is no general theory here, but we provide some worked-out examples that show how the orbit method can suggest the right answer and give a visual and adequate description of the situation.

1. Semisimple groups

In a sense, this class of Lie groups is opposite to the case of nilpotent groups, hence, the application of the orbit method is a complicated problem.[1]

1.1. Complex semisimple groups.

This is the most well-understood class of semisimple groups. The status of the representation theory for these groups can be found in surveys [**Bar**], [**BV**], and [**Du2**].

The application of the orbit method for these groups was suggested in my lectures at Institut Henri Poincaré (Paris, 1968). I observed that the so-called principal series of unitary representations is in perfect correspondence with closed coadjoint orbits of maximal dimension. Indeed, it is known that an adjoint orbit $\Omega \subset \mathfrak{g}$ is closed iff the stabilizer of a point $X \in \Omega$ is reductive (then the element X itself is ad-semisimple; see, e.g., [**Bor**]).

In the notes of these lectures written by M. Duflo he included his own important result: the integral formula for the generalized characters (i.e.

[1] Note, however, that for a quantum deformation of a semisimple group the dual quantum group is solvable. So, there is another argument in favor of the orbit method (see [**So**] for details).

Rule 6 of the User's Guide) is true for all representations of the principal series (see [**Du1**]).

One can associate the so-called **degenerate series** with degenerate orbits (i.e. closed orbits of lower dimensions). Indeed, these representations are induced from 1-dimensional representations of parabolic subgroups of G in full accordance with Rule 2.

As for non-closed orbits, the most interesting of them are the so-called nilpotent orbits consisting of ad-nilpotent elements. There is a rich and difficult theory of representations that could be associated with nilpotent orbits; see [**Vo2**].

Finally, there are the so-called **complementary series** of unirreps that are usually constructed using the analytic continuation of principal or degenerate series. It seems that they can be associated with coadjoint orbits in the complexification $\mathfrak{g}^*_{\mathbb{C}}$ of $\mathfrak{g}^*$. We say more about this in Section 6.4.

1.2. Real semisimple groups.

This class of groups is the hardest for representation theorists. It is enough to say that the unitary duals of general real semisimple groups remain unknown despite more than half a century of effort. (The first result, the Bargmann description of the unitary dual for $SL(2, \mathbb{R})$, appeared in 1947.)

The deepest results here are due to Harish-Chandra and Langlands, but the theory is still far from complete. For surveys see [**Vo1**], [**Wa**], and [**Zh2**].

We observe that some basic facts from the representation theory of real semisimple groups can be easily described (but not proved) in terms of coadjoint orbits. Consider, for example, the famous Harish-Chandra theorem about discrete series. The theorem claims that a real semisimple group G possesses a discrete series of representations iff it has a compact Cartan subgroup.

This can be explained by the geometry of orbits as follows. A closed orbit of maximal dimension has the form $\Omega = G/H$, where H is a Cartan subgroup in G. The number of integrality conditions for Ω is equal to $b_2(\Omega) = b_1(H)$. This number coincides with the dimension of the orbit space (which is $\operatorname{rk} G = \dim H$) only when H is compact.

2. Lie groups of general type

It is known that a general Lie group G can be written in the form of a semidirect product $G = S \ltimes R$ where S is a maximal semisimple subgroup and R is a maximal solvable normal subgroup. We describe below two examples of this type of group.

2.1. Poincaré group.

This group is well known because it is the symmetry group of the Minkowski space, the main object of special relativity theory.

Recall that Minkowski space $M = \mathbb{R}^{1,3}$ is a pseudo-Euclidean space with the quadratic form

$$Q(x) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \qquad \text{or} \qquad c^2t^2 - x^2 - y^2 - z^2.$$

The Poincaré group is the group of rigid motions of M, that is, the semidirect product $P = O(1, 3, \mathbb{R}) \ltimes \mathbb{R}^{1,3}$. We denote by P_0 the subgroup $SO_0(1, 3, \mathbb{R}) \ltimes \mathbb{R}^{1,3}$, which consists of transformations preserving the space and time orientations. This group is connected but not simply connected.

The simply connected covering group $\widetilde{P}_0$ has a convenient matrix realization by complex block matrices of the form

$$p = p_0 p_1, \qquad p_0 = \begin{pmatrix} g^{-1} & 0 \\ 0 & g^* \end{pmatrix}, \qquad p_1 = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}), \quad h = h^* = \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{C}).$$

The Lie algebra $\mathfrak{p}$ consists of complex block matrices

$$X = \begin{pmatrix} -a & h \\ 0 & a^* \end{pmatrix}, \qquad a \in \mathfrak{sl}(2, \mathbb{C}),\ h = h^* \in \mathrm{Mat}_2(\mathbb{C}).$$

The space of 2×2 Hermitian matrices is identified with M via the coordinates x^0, x^1, x^2, x^3 above. The action of the group $\widetilde{P}_0$ on M comes from the ordinary matrix multiplication and has the form

$$p \cdot x = p_0 p_1 \cdot x = g^{-1}(x + h)(g^*)^{-1}.$$

We see that p_0 acts as a translation and p_1 as a pseudo-rotation in Minkowski space because the quadratic form $\det x = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = Q(x)$ is preserved.

Exercise 1. Show that the unitary matrices from $SU(2) \subset SL(2, \mathbb{C})$ act as rotations in M and Hermitian matrices act by **Lorentz transformations**. ♣

The space $\mathfrak{p}^*$, dual to $\mathfrak{p}$, can be identified with the space of block matrices of the form

$$F = \begin{pmatrix} -b^* & 0 \\ c & b \end{pmatrix} \qquad \text{with } \langle F, X \rangle = \mathrm{tr}\,(FX) = 2\mathrm{Re}\,\mathrm{tr}\,(ba^*) + \mathrm{tr}\,(ch).$$

We write the coadjoint action separately for rotations and for translations:

$$K(p_0)(b, c) = \big(g^{-1}bg,\, g^*cg\big), \qquad K(p_1)(b, c) = \big((b - hc)_0,\, c\big)$$

where $(x)_0$ denotes the traceless part of x.

Assume that $Q(h) = \det h > 0$. Then, using an appropriate rotation, we can reduce the Hermitian matrix c to the scalar form $c = m \cdot 1$ with some $m > 0$.

Thereupon, we can apply a translation so that c remains the same and we remove from b its Hermitian part. As a result, b becomes an anti-Hermitian matrix with zero trace.

Finally, using a unitary rotation, we preserve c and reduce b to the diagonal form $b = \begin{pmatrix} is & 0 \\ 0 & -is \end{pmatrix}$. So, the orbits with $Q(c) > 0$ are parametrized by two numbers $m > 0$ and s. A more detailed consideration shows that the orbit in question is subjected to one integrality condition: the number s must belong to $\frac{1}{2}\mathbb{Z}$. In physical applications the parameters (m, s) are interpreted as the rest-mass (or energy) and the spin of a particle. $\diamondsuit$

2.2. Odd symplectic groups.

The groups $Sp(2n + 1, \mathbb{R})$, the so-called **odd symplectic groups**, were introduced independently by many authors (see [**Pr**]). We define $G = Sp(2n + 1, \mathbb{R})$ as a stabilizer of the first basic vector in the standard realization of $Sp(2n+2, \mathbb{R})$ in $\mathbb{R}^{2n+2}$. This group is a semidirect product of $Sp(2n, \mathbb{R})$ and the generalized Heisenberg group H_n. The unirreps of this group were recently studied in [**BS**].

The Lie algebra $\mathfrak{g}$ consists of matrices of order $2n+2$ that have the block form

$$\begin{pmatrix} 0 & a^t & b^t & c \\ 0 & A & B & b \\ 0 & C & -A^t & -a \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$a, b \in \mathbb{R}^n, \quad c \in \mathbb{R}, \quad A, B, C \in \mathrm{Mat}_n(\mathbb{R}), \quad B = B^t,\ C = C^t.$$

It also has a simple realization in terms of canonical commutation relations. Let $p_1, \dots, p_n, q_1, \dots, q_n$ be the canonical operators satisfying CCR (see Chapter 2):

$$[p_k,\, p_j] = [q_k,\, q_j] = 0, \qquad [p_k,\, q_j] = \frac{h}{2\pi i} \cdot \delta_{kj}.$$

Consider the space P_2 of all polynomials in p_i, q_j of degree ≤ 2. It is easy to check that P_2 is a Lie algebra with respect to the commutator $[a, b] = ab - ba$. I claim that this Lie algebra is isomorphic to $\mathfrak{g}$.

Indeed, the correspondence

$$\begin{pmatrix} 0 & a^t & b^t & c \\ 0 & A & B & b \\ 0 & C & -A^t & -a \\ 0 & 0 & 0 & 0 \end{pmatrix} \mapsto \frac{i}{2\hbar}(-\hbar \ \ -q^t \ p^t \ 1) \begin{pmatrix} 0 & a^t & b^t & c \\ 0 & A & B & b \\ 0 & C & -A^t & -a \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ p \\ q \\ \hbar \end{pmatrix}$$

$$= i\big(ap^t + bq^t\big) + ic\hbar + \frac{1}{2i\hbar}\big(q^t Ap + p^t A^t q + q^t Bq + p^t Cp\big)$$

establishes the desired isomorphism of $\mathfrak{g}$ onto P_2.

In particular for $n = 1$ we get the following representation of $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{h}$:

$$X \mapsto ip, \quad Y \mapsto iq, \quad Z \mapsto i\hbar, \quad E \mapsto \frac{iq^2}{2\hbar}, \quad H \mapsto \frac{i(pq+qp)}{2\hbar}, \quad F \mapsto -\frac{ip^2}{2\hbar}.$$

Note that the central element c of $\mathfrak{h}_n$ (corresponding to the monomial 1) also belongs to the center of the whole Lie algebra $\mathfrak{g}$.

The dual space $\mathfrak{g}^*$ is realized by matrices of the block form

$$F = \begin{pmatrix} * & * & * & * \\ x & R & P & * \\ y & Q & -R^t & * \\ z & y^t & -x^t & * \end{pmatrix},$$

$$x, y \in \mathbb{R}^n, \quad z \in \mathbb{R}, \quad P,\, Q,\, R \in \mathrm{Mat}_n(\mathbb{R}), \quad P = P^t,\ Q = Q^t,$$

where the asterisks remind us that F is not a matrix but an element of the factor-space $\mathrm{Mat}_{2n+2}(\mathbb{R})/\mathfrak{g}^\perp$.

The most interesting representations of G correspond to orbits Ω_h that contain an element F with all entries zero except $z = h \neq 0$. A direct computation shows that Ω_h consists of matrices of the form

$$F = \begin{pmatrix} * & * & * & * \\ x & -xh^{-1}y^t & -xh^{-1}x^t & * \\ y & yh^{-1}y^t & yh^{-1}x^t & * \\ h & y^t & -x^t & * \end{pmatrix}.$$

It is clear that the projection of Ω_h to $\mathfrak{h}_n^*$ is a single H_n-orbit corresponding to the irreducible representation π_h of H_n (see Chapter 2).

On the other hand, the projection of Ω_h to $\mathfrak{sp}(2n, \mathbb{R})^*$ is also a single orbit of dimension $2n$ that consists of matrices of rank 1. Actually, this projection depends only on the sign of h, so, we denote it by $\Omega_\pm$. The orbit Ω_+ consists of all matrices F for which FJ_{2n} is a non-negative symmetric matrix of rank 1 and $\Omega_- = -\Omega_+$.

Both orbits $\Omega_\pm$ for $n > 1$ admit no $Sp(2n, \mathbb{R})$-invariant polarization, neither real, nor complex. Therefore, Rule 2 of the User's Guide does not work in this situation.

But according to Rule 3 of the User's Guide, we can try to construct the representations corresponding to $\Omega_\pm$ by extending the representation $\pi_{\pm 1}$ of H_n to $Sp(2n+1, \mathbb{R})$ and then restrict it to $Sp(2n, \mathbb{R})$. This can be done by expressing the elements of $\mathfrak{g}$ in terms of canonical operators $p_1, \ldots, p_n; q_1, \ldots, q_n$. Note that this method gives us the same expression for both orbits $\Omega_\pm$.

The resulting representation π of $\mathfrak{sp}(2n, \mathbb{R})$ is called the **metaplectic representation**. It gives rise to a double-valued representation of $Sp(2n, \mathbb{R})$ which becomes single-valued on the two-sheeted covering group $Mp(2n, \mathbb{R})$, the so-called **metaplectic group**.

We consider in more detail the simplest case $n = 1$, $\hbar = 1$. For the canonical basis of $\mathfrak{sl}(2, \mathbb{R})$ we get the following expressions:

$$\pi(E) = \frac{q^2}{2i} = \frac{ix^2}{2}, \ \pi(F) = -\frac{p^2}{2i} = \frac{1}{2i}\frac{d^2}{dx^2}, \ \pi(H) = \frac{i(pq+qp)}{2} = x\frac{d}{dx} + \frac{1}{2}.$$

Consider the element $X = E - F$ that generates a 1-parametric compact subgroup $\exp tX$ in G with period 2π. We have $\pi(X) = \frac{i}{2}\big(x^2 - \frac{d^2}{dx^2}\big)$. As we have seen in Chapter 2, this operator has a point spectrum consisting of eigenvalues $\frac{i}{2}, \frac{3i}{2}, \frac{5i}{2}, \ldots$. In particular, $\pi\big(\exp(2\pi X)\big) = e^{2\pi\cdot\pi(X)} = -1$. Therefore, the representation in question is correctly defined only on $Mp(2n, \mathbb{R})$.

3. Beyond Lie groups

In this section we only briefly describe the application of the orbit method beyond Lie groups. We refer to original papers for more details.

3.1. Infinite-dimensional groups.

Some infinite-dimensional groups can be viewed as infinite-dimensional smooth manifolds. So, they have a tangent space and adjoint and coadjoint representations. The correspondence between coadjoint orbits and unirreps is more complicated than in the finite-dimensional case, but it still exists.

Example 1. Consider the infinite unitary group. While for a finite n there is only one unitary group $U(n) := U(n, \mathbb{C})$, there are several different candidates for the role of $U(\infty)$. Let H be a Hilbert space of Hilbert dimension $\aleph_0$ with an orthonormal basis $\{x_k\}_{1 \le k < \infty}$.

The maximal unitary group $U(\infty)$ consists of all unitary operators in H. The minimal one is the group $U_0(\infty) := \bigcup_{n \ge 1} U(n)$. It is the subgroup of $U(\infty)$ consisting of operators that fix all but finitely many basic vectors.

There are many intermediate subgroups between these two. In particular, we have a chain of groups

$$U_0(\infty) \subset U_f(\infty) \subset \cdots \subset U^{(p)}(\infty) \subset \cdots \subset U_c(\infty) \subset U(\infty)$$

where all groups are defined by conditions on the operator $a = u - 1$.

Namely:

for $u \in U_f(\infty)$: a has a finite rank;

for $u \in U^{(p)}(\infty)$, $1 \leq p < \infty$: a belongs to the **Shatten ideal** $L^p(H)$ given by the condition $\operatorname{tr}(a^*a)^{p/2} < \infty$;

for $u \in U_c(\infty)$: a is a compact operator.

The corresponding tangent space at the unit can be defined for all these groups using Stone's theorem about 1-parametric groups of unitary operators. Namely:

$\mathfrak{u}_0$ is the space of skew self-adjoint operators annihilating all but finitely many basic vectors;

$\mathfrak{u}_f(\infty)$ is the space of skew self-adjoint operators of finite rank;

$\mathfrak{u}^{(p)}(\infty)$ for $1 \leq p < \infty$ is the space of skew self-adjoint operators from $L^p(H)$;

$\mathfrak{u}_c(\infty)$ is the space of skew self-adjoint compact operators;

$\mathfrak{u}(\infty)$ is the set of skew self-adjoint (not necessarily bounded) operators.

Note that the set $\mathfrak{u}(\infty)$ is not a Lie algebra and is not even a vector space. All other tangent spaces have a natural Lie algebra structure defined by the formula $[a, b] = ab - ba$.

The dual space to $\mathfrak{u}^{(p)}(\infty)$ for $p > 1$ is identified with $\mathfrak{u}^{(q)}(\infty)$, $q = \frac{p}{p-1}$, while $\mathfrak{u}^{(1)}(\infty)^*$ is the space of all bounded skew self-adjoint operators. The elements of $\mathfrak{u}_0^*$ can be realized as infinite skew self-adjoint matrices with no restriction on matrix entries.

From the subgroups listed above only one is closed in the norm topology: the group $U_c(\infty)$.

Let us think about what kind of prediction can be made about the unirreps of $U_c(\infty)$ using the ideology of the orbit method. According to Rule 1 of the User's Guide, the unirreps of G are related to integral coadjoint orbits in $\mathfrak{g}$ (possibly with additional data if the orbit is not simply connected).

The classification of integral coadjoint orbits for $U_c(\infty)$ can be easily achieved. The point is that every linear functional on the space of skew self-adjoint compact operators, which is continuous in the norm topology,

has the form

$$f(a) = \operatorname{tr}(ab) \quad \text{where } b \text{ is a skew self-adjoint operator of trace class.}$$

The orbit Ω_b passing through the skew self-adjoint operator $b \in L^1(H)$ is integral iff the spectrum of b is contained in $i\mathbb{Z}$. For an operator of trace class this is possible only if b has finite rank. Then it can be reduced by the coadjoint action of $U_c(\infty)$ to the diagonal form with non-zero entries $i\mu_j$, $i\nu_k$, satisfying

$$\nu_n \leq \cdots \leq \nu_1 < 0 < \mu_1 \leq \cdots \leq \mu_m. \tag{1}$$

So, the orbit method predicts that unirreps of $U_c(\infty)$ are labelled by sequences of the form (1). And this is indeed so!

To show this, we recall the definition of the **Schur functor** $S^{(\lambda)}$ in the category of vector spaces (see, e.g., [**FH**], §6.1). Let λ be an N-tuple $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0) \in \mathbb{Z}^N$ where N can be arbitrary. Denote by $S^k(V)$ the k-th symmetric power of V with an additional agreement: $S^0(V) = \mathbb{C}$, $S^k(V) = \{0\}$ for $k < 0$.

Now we define the functor $S^{(\lambda)}$ by the formula

$$S^{(\lambda)}(V) = \det \|S^{\lambda_i + j - i}(V)\|_{1 \leq i,j \leq N}. \tag{2}$$

Instead of the formal definition, we explain how to understand the right-hand side of (2) with examples:

$$S^{1,1}(V) = \det \begin{pmatrix} S^1(V) & S^2(V) \\ S^0(V) & S^1(V) \end{pmatrix} = V \otimes V - S^2(V) = \bigwedge^2(V);$$

$$\begin{aligned} S^{2,1,1}(V) &= \det \begin{pmatrix} S^2(V) & S^3(V) & S^4(V) \\ S^0(V) & S^1(V) & S^2(V) \\ S^{-1}(V) & S^0(V) & S^1(V) \end{pmatrix} \\ &= S^2(V) \otimes V \otimes V + S^4(V) - S^2(V) \otimes S^2(V) - S^3(V) \otimes V \\ &= S^2(V) \otimes \bigwedge^2(V) - \big(S^3(V) \otimes V - S^4(V)\big). \end{aligned}$$

In these expressions the difference $W_1 - W_2$ is understood as a $GL(V)$-module W_3 such that $W_1 \simeq W_2 \oplus W_3$. It is a good (but not trivial) exercise to check that the $GL(V)$-module $S^{(\lambda)}(V)$ is simple and that the corresponding irreducible representation of $GL(V)$ is precisely the representation π_λ with the highest weight λ.

The unirrep $\pi_{\mu;\nu}$ of $U_c(\infty)$ corresponding to the label (1) acts naturally in the space $S^\mu(H) \boxtimes S^\nu(\overline{H})$ where $\overline{H}$ is the dual space to H.

Notice that for the finite-dimensional group $U(n)$ this representation is generally reducible, while for $U_c(\infty)$ it is always irreducible. To understand this phenomenon, look at $\pi_{1;1}$. The representation space is $V \boxtimes V^*$. In the finite-dimensional case this is identified with $\mathrm{End}(V)$ and splits into a scalar part and a traceless part. In the infinite-dimensional case it is identified with the space of Hilbert-Schmidt operators in H and does not contain a scalar part. $\diamondsuit$

Example 2. Kac-Moody algebras and Kac-Peterson groups.

A very interesting example of infinite-dimensional algebras was discovered by V. Kac and R. Moody. They form a natural generalization of semisimple Lie algebras related to extended Dynkin diagrams. The corresponding class of infinite-dimensional Lie groups was introduced by V. Kac and D. H. Peterson. We briefly describe this example referring to [**Ka1**], [**KP**], and [**PS**] for the details.

Let $L(K)$ be the **loop group** of all smooth maps from the circle S^1 to a compact simply connected Lie group K with pointwise multiplication. It is a connected and simply connected topological group, since $\pi_1(K) = \pi_2(K) = \{0\}$. Moreover, it is an infinite-dimensional Lie group with Lie algebra $L(\mathfrak{k}) = C^\infty(S^1, \mathfrak{k})$.

Let $\{X_1, \ldots, X_n\}$ be an orthonormal basis in $\mathfrak{k}$ with respect to the negative Killing form, with the structure constants c_{ij}^k. Then the functions

$$f_{m,k}(t) = e^{2\pi imt} \cdot X_k, \quad 1 \leq k \leq n, \ m \in \mathbb{Z},$$

form a complete orthonormal system with respect to the Ad-invariant bilinear form in $L(\mathfrak{k})$ given by

$$\big(f(\cdot), g(\cdot)\big) = -\int_0^1 \big(f(t), g(t)\big)_K dt.$$

This Lie algebra has a non-trivial central extension given by the cocycle

$$c(f_{m,k}, f_{m',k'}) = m\delta_{m,-m'}\delta_{k,k'} \quad \text{or} \quad c(f, g) = -\int_0^1 \big(f(t), g'(t)\big)_K dt.$$

The description of coadjoint orbits in this case is equivalent to the Floquet theory for an ordinary differential equation with periodic coefficients. The representations of Kac-Moody algebras and Kac-Peterson groups have been intensively studied, especially by mathematical physicists. We refer to [**EFK**], [**FF2**], [**GO**], and the vast literature on the web.

It turns out that there is a good correspondence between orbits and unirreps. The most impressive result was obtained in [**Fr**] where the integral

formula for the characters was obtained, which is the direct analogue of Rule 7 of the User's Guide. ♢

Example 3. Virasoro-Bott group Vir.

This is an interesting example that is related to many physical applications.

Let $G = \mathrm{Diff}_+(S^1)$ be the group of orientation preserving diffeomorphisms of the circle. The simply connected covering $\widetilde{G}$ of this group can be realized by orientation preserving diffeomorphisms of the real line, which commute with integral translations. Such a diffeomorphism has the form

$$x \mapsto \phi(x), \qquad \phi'(x) > 0, \qquad \phi(x+n) = \phi(x) + n, \quad n \in \mathbb{Z},$$

where the function ϕ is determined modulo the integer summand. The Lie algebra of this infinite-dimensional Lie group is just the space $Vect^\infty(S_1)$ of smooth vector fields on the circle, which can be identified with $C^\infty(S^1)$ by $v(x)\frac{d}{dx} \leftrightarrow v(x)$.

The group Vir is a non-trivial central extension of the group $\widetilde{G}$. On the level of Lie algebras this extension is given by a cocycle discovered around 1970 independently by I.M. Gelfand – D.B. Fuchs and M. Virasoro:[2]

$$c(v,\, w) = \int_0^1 v'(x) dw'(x).$$

On the group level the cocycle was first computed by R. Bott and is

$$c(\phi,\, \psi) = \int_0^1 \log(\phi \circ \psi)' d\log(\psi').$$

There is a vast literature devoted to the coadjoint orbits of this group (see, e.g., [**Ki12**], [**Wi2**]) and its representations, starting with the famous article [**BPZ**], where the representations of the Lie algebra vir were related to conformal field theory and to some integrable problems in statistical mechanics. The mathematical approach can be found in [**FF2**], [**GO**], [**KY**], [**Ner**].

Unfortunately, this subject is too big to be described here. ♢

3.2. $\mathfrak{p}$-adic and adelic groups.

Another possibility to extend the orbit method is connected with the existence of fields different from $\mathbb{R}$ and $\mathbb{C}$. The experience of commutative harmonic analysis shows that the most promising is the case of locally compact non-discrete topological fields. Such fields are completely classified.

[2]In pure algebraic form this cocycle was discovered earlier by R. Block.

Those that have characteristic zero (i.e. contain the simple subfield $\mathbb{Q}$) are just algebraic extensions of the p-adic fields discussed in Appendix I.1.2. They are called $\mathfrak{p}$-adic fields and play an important role in modern number theory.

Still more interesting is the ring of adeles $\mathbb{A}(K)$ invented by A. Weil for the study of number fields K (finite extensions of $\mathbb{Q}$). It appears naturally if you want to describe the Pontrjagin dual to the additive group of K with discrete topology. For $K = \mathbb{Q}$ the ring of adeles $\mathbb{A}(\mathbb{Q})$, or simply $\mathbb{A}$, can be defined as a subset of the direct product $\mathbb{R} \times \prod_{p \text{ prime}} \mathbb{Q}_p$ that consists of sequences $\{a_p\}$, p prime or ∞, where $a_p \in \mathbb{Q}_p$, $a_\infty \in \mathbb{R}$, and $a_p \in \mathbb{Z}_p$ for all but a finite number of primes.

The field $\mathbb{Q}$ is "diagonally" embedded in $\mathbb{A}$ in the form of "constant" sequences $\{r, r, r, \dots\}$ where the first r is considered as a real rational number, the second as a 2-adic number, etc.

Proposition 1. *The exact sequence* $0 \longrightarrow \mathbb{Q} \longrightarrow \mathbb{A} \longrightarrow \mathbb{A}/\mathbb{Q} \longrightarrow 0$ *is Pontrjagin self-dual:*

$$\widehat{\mathbb{Q}} \simeq \mathbb{A}/\mathbb{Q}, \qquad \widehat{\mathbb{A}} \simeq \mathbb{A}, \qquad \widehat{\mathbb{A}/\mathbb{Q}} \simeq \mathbb{Q}.$$

□

For any algebraic group G defined over a number field K one can define the groups $G\big(\mathbb{A}(K)\big)$ and $G(K)$ so that the latter group is a discrete subgroup in the former one. Moreover, it is known that on the homogeneous space $X(G, K) = G\big(\mathbb{A}(K)\big)/G(K)$ there is a canonically defined invariant measure τ. So, we get a unitary representation π of $G\big(\mathbb{A}(K)\big)$ in the space $L^2\big(X(G, K), \tau\big)$. The decomposition of π into unirreps is a very important and difficult problem in modern number theory.

The orbit method suggests that the spectrum of π is related to the "rational" orbits in $\mathfrak{g}^*\big(\mathbb{A}(K)\big)$, i.e. those that contain a point from $\mathfrak{g}^*(K)$. I recommend that the interested reader consider in detail the simplest case $G = H$, the Heisenberg group. I refer to [**Mo**] and [**Ho**] for some results in this direction.

A much more difficult and intriguing case of semisimple algebraic groups over $\mathbb{A}$ is related to the so-called Langlands program. But here we enter the realm of arithmetic geometry and I must stop and refer to experts in this very deep theory.

3.3. Finite groups.

Some finite groups can be viewed as algebraic groups over a finite field $\mathbb{F}_q$ with q elements. Therefore, one can try to apply the orbit method to these groups and relate the unirreps of G to the coadjoint orbits in $\mathfrak{g}^*$.

One of the first candidates for this approach is the group N_n of upper triangular matrices with units on the main diagonal. We consider N_n as an algebraic subgroup in $GL(n)$ given by the equations

$$g_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i > j. \end{cases} \tag{3}$$

For any field K we denote by $N_n(K)$ the group of K-points of N_n, i.e. the group of upper triangular matrices with elements from K satisfying (3). The Lie algebra $\mathfrak{n}_n(K)$ is defined as a tangent space to the algebraic manifold $N_n(K)$ and consists of strictly upper triangular matrices with elements from K. We can also define the dual space $\mathfrak{n}_n^*$ and the coadjoint action of $N_n(K)$ on it.

Unfortunately, the classification problem for $N_n(K)$-orbits in the space $\mathfrak{n}^*$ is still open despite many attempts to solve it.

It was understood long ago that this problem practically does not depend on the field K. For $n \leq 6$, where the full classification is known, the set $\mathcal{O}_n$ of coadjoint orbits of N_n in $\mathfrak{n}_n^*$ is a union of simple "cells" Ω_i of different dimensions d_i, which are quasiaffine manifolds of type $K^n \backslash K^m \cong \mathbb{A}^m(K) \times GL(1, K)^{n-m}$. It seems that a similar, or slightly more complicated, picture exists in higher dimensions too.

In particular, it is natural to conjecture that for a finite field $\mathbb{F}_q$ of q elements the number of orbits is a polynomial in q.[3]

Moreover, if this conjecture is true and if we denote by $\mathcal{O}_n^m(\mathbb{F}_q)$ the set of all $2m$-dimensional orbits[4] in $\mathfrak{n}_n^*(\mathbb{F}_q)$, then the sum

$$P_n(q, t) = \sum_{m\geq 0} t^m \cdot \#\mathcal{O}_n^m(\mathbb{F}_q)$$

must be a polynomial in q and t. This polynomial encodes all essential information about the set of orbits.

For small n we have:

$$\begin{aligned} P_0 &= 1, \\ P_1 &= 1, \\ P_2 &= q, \\ P_3 &= (q-1)t + q^2, \\ P_4 &= q(q-1)t^2 + (q-1)q(q+1)t + q^3, \\ P_5 &= (q-1)^2 t^4 + (q-1)q(2q-1)t^3 \\ &\quad + (q^2-1)q(2q-1)t^2 + (q-1)q^2(2q+1)t + q^4. \end{aligned}$$

[3] Recently this problem was solved by I. M. Isaacs (see [I]).

[4] Recall that all coadjoint orbits are symplectic manifolds, hence have an even dimension.

It was shown in [**Ki10**] that the beginning terms of the sequence $\{P_n\}$ written above (and even more detailed information about the orbits) can be obtained by a variant of the so-called **Euler-Bernoulli triangle** (cf. [**Ar1**]). But in the seventh row of the triangle a discrepancy arises. So, the general formula for P_n is still unknown.

Note that polynomials $Y_n(q) := P_n(q, q)$ have a still nicer behavior. To describe it we give here the definition of four remarkable sequences of polynomials that were introduced in [**KM**]:

$$\{A_n\}, \quad \{B_n\}, \quad \{C_n\}, \quad \{D_n\}, \quad n \geq 0, \quad \text{in} \quad \mathbb{Z}[q].$$

1. $A_n(q)$ is by definition the number of solutions to the equation $X^2 = 0$ in the space of $n \times n$ upper-triangular matrices with elements from $\mathbb{F}_q$. More precisely, let $A_n^r(q)$ be the number of solutions that have rank r. These quantities satisfy the simple recurrence relations

$$A_{n+1}^{r+1}(q) = q^{r+1} \cdot A_n^{r+1}(q) + (q^{n-r} - q^r) \cdot A_n^r(q), \qquad A_{n+1}^0(q) = 1, \quad n \geq 0.$$

We see, in particular, that $A_n^r(q)$ are polynomials in q. Hence, $A_n(q) = \sum_{r \geq 0} A_n^r(q)$ are polynomials as well.

2. Consider the triangle

$$\begin{array}{ccccccccc}
 & & & & b_{00} & & & & \\
 & & & b_{10} & \longleftarrow & b_{01} & & & \\
 & & b_{02} & \longrightarrow & b_{11} & \longrightarrow & b_{20} & & \\
 & b_{30} & \longleftarrow & b_{21} & \longleftarrow & b_{12} & \longleftarrow & b_{03} & \\
b_{04} & \longrightarrow & b_{13} & \longrightarrow & b_{22} & \longrightarrow & b_{31} & \longrightarrow & b_{40} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}$$

formed by polynomials $b_{k,l}(q)$, $k \geq 0, l \geq 0$, which are labelled in a "shuttle way" and satisfy the recurrence relations

$$\begin{aligned} b_{k,l} =& q^{-1} b_{k-1,\, l+1} + (q^{l+1} - q^l) b_{l,\, k-1} \text{ for } k > 0; \\ b_{0,l} =& q^l b_{l-1,0} \text{ for } l > 0; \quad b_{0,0} = 1. \end{aligned}$$

Now put $B_n(q) := b_{n-1,0}(q)$, $n > 0$, $B_0(q) = 1$. This is our second sequence.

3. Define the generalized Catalan numbers $c_{k,l}$ for $k \geq 1$, $|l| \leq k$, $k \equiv l \mod 2$, by

$$c_{k,k-2s} = \binom{k-1}{s} - \binom{k-1}{s-1}.$$

It is convenient to set $c_{k,l} = 0$ for $|l| > k$. The triangle formed by these numbers satisfies the same recurrence relation as the Pascal triangle: $c_{k,l} =$

$c_{k-1,l} + c_{k,l-1}$, but with a different initial condition: $c_{1,-1} = -1$, $c_{1,1} = 1$ instead of $c_{0,0} = 1$.

The numbers $c_n := c_{2n+1,1}$, $n \geq 0$, form the sequence of ordinary Catalan numbers:[5] 1, 1, 2, 5, 14, 42, 132, ...

It is pertinent to remark that for a positive l the generalized Catalan number $c_{k,l}$ equals the dimension of the irreducible representation of the symmetric group S_{k-1} corresponding to the partition $(2^{\frac{k-l}{2}}, 1^{l-1})$. In particular, the ordinary Catalan number c_n corresponds to the rectangular diagram (2^n).

The third sequence is defined by the formula

$$C_n(q) = \sum_s c_{n+1,s} \cdot q^{\frac{n^2}{4} + \frac{1-s^2}{12}}$$

where the sum is taken over all integers $s \in [-n-1, n+1]$ that satisfy

$$s \equiv n+1 \pmod 2, \qquad s \equiv (-1)^n \pmod 3.$$

As was observed in [**KM**], the first 26 polynomials $C_n(q)$ coincide with $A_n(q)$ and $B_n(q)$. Later it was shown by Doron Zeilberger, using the technique from [**PWZ**], that $A_n = C_n$ for all $n \geq 0$.

4. The fourth sequence is defined by

$$D_n(q) := \zeta_{N_n(\mathbb{F}_q)}(-1) = \sum_{\lambda \in \widehat{N}_n(\mathbb{F}_q)} d(\lambda).$$

Here $d(\lambda)$ is the dimension of a unirrep of class λ and $\zeta_G(s) = \sum_{\lambda \in \widehat{G}} d(\lambda)^{-s}$ is the so-called group zeta-function of the group G (see, e.g., [**Ki9**]). In other words, $D_n(q)$ is the sum of the dimensions of all irreducible representations of the finite group $N_n(\mathbb{F}_q)$.

Now we can formulate the remarkable property of the sequence $\{Y_n\}$.

Proposition 2 (see [**KM**]). *For $0 \leq n \leq 11$ we have*

$$A_n(q) \equiv B_n(q) \equiv C_n(q) \equiv D_n(q) \equiv Y_n(q).$$

[5]The Catalan numbers are usually defined by the recurrence $c_{n+1} = \sum_{k=0}^{n} c_k \cdot c_{n-k}$ and the initial conditions $c_0 = c_1 = 1$.

Here are the first dozen of these polynomials:

$$\begin{aligned}
A_0 &= 1,\\
A_1 &= 1,\\
A_2 &= q,\\
A_3 &= 2q^2 - q,\\
A_4 &= 2q^4 - q^2,\\
A_5 &= 5q^6 - 4q^5,\\
A_6 &= 5q^9 - 5q^7 + q^5,\\
A_7 &= 14q^{12} - 14q^{11} + q^7,\\
A_8 &= 14q^{16} - 20q^{14} + 7q^{12},\\
A_9 &= 42q^{20} - 48q^{19} + 8q^{15} - q^{12},\\
A_{10} &= 42q^{25} - 75q^{23} + 35q^{21} - q^{15},\\
A_{11} &= 132q^{30} - 165q^{29} + 44q^{25} - 10q^{22}.
\end{aligned}$$

To finish the section, we list some problems here that look very intriguing and are still unsolved.

1. Give the description of coadjoint orbits of the group N_n over an arbitrary field.

2. Describe the asymptotics of the number of coadjoint orbits for the group $N_n(\mathbb{F}_q)$ when n goes to infinity. In particular, compute

$$\limsup_{n\to\infty} \frac{\log\log \# \mathcal{O}_n(\mathbb{F}_q)}{\log n}.$$

3. Prove (or disprove) that the five sequences $A_n(q)$, $B_n(q)$, $C_n(q)$, $D_n(q)$, $Y_n(q)$ coincide for all $n \geq 0$.

4. The most intriguing problem is to interpret the formula for $\# O_n(\mathbb{F}_q)$ given in [**Ki10**]:

$$\# O_n(\mathbb{F}_q) = \frac{1}{\# N_n(\mathbb{F}_q)} \sum_{\substack{F\in\mathfrak{n}^*(\mathbb{F}_q)\\ X,Y\in\mathfrak{n}(\mathbb{F}_q)}} \theta(\langle F, [X, Y]\rangle)$$

as the partition function of some quantum-mechanical system. Note that in the natural generalization of this formula to algebraic extensions of $\mathbb{F}_q$ the degree of the extension plays the role of the standard parameter β (inverse temperature) in statistical mechanics.

3.4. Supergroups.

Up to now we have not mentioned one remarkable extension of the notions of smooth manifold and Lie group. This extension is a part of the general idea of supersymmetry advanced by physicists to explain the analogy between bosonic and fermionic particles.

In mathematics supersymmetry requires equal rights for plus and minus, for even and odd, and for symmetric and antisymmetric. In short, the ideology of supersymmetry means the following. To any "ordinary" (or "even") notion, definition, theorem, etc. there corresponds an "odd" analogue which together with the initial notion form a "superobject".

The formalism of supersymmetry requires a new sort of numbers that differ from the ordinary ones by the property of a product. It is not commutative but anticommutative: $xy = -yx$; in particular, $x^2 = 0$. This new sort of numbers must be used as widely as the ordinary numbers. For example, they can play the role of local coordinates on a manifold. Thus, the notion of a supermanifold arises.

We refer to [**Le**], [**Mn1**], [**Mn2**], [**QFS**], [**Wi1**] for more details.

Here we only want to say that for the so-called **supergroups**, which are group-like objects in the category of supermanifolds, all ingredients of the orbit method make sense. Moreover, for real nilpotent supergroups there is a perfect correspondence between unirreps and even coadjoint orbits (see [**Ka2**]).

There are many papers in the physical literature where unirreps of "classical" supergroups $SL(m|n, \mathbb{R})$ and $OSp(k|2n, \mathbb{R})$ are considered. It would be interesting to relate these unirreps to coadjoint orbits.

4. Why the orbit method works

Some people consider the orbit method a miracle that cannot be explained by natural arguments. Nevertheless, there exist two independent "explanations" of this phenomenon. We discuss these arguments below.

Note, however, that both these explanations fail in the case of finite or p-adic groups, so at least a part of the miracle still remains.

4.1. Mathematical argument.

The idea behind this argument goes back to the following simple but important observation. Let G be a matrix Lie group, which can be a rather complicated curved submanifold $G \subset \mathrm{Mat}_n(\mathbb{R})$. The **logarithm map**

$$G \ni g \mapsto \log g := \sum_{k \geq 1} \frac{(-1)^{k-1}(g-1)^k}{k} \tag{4}$$

transforms G into a linear subspace $\mathfrak{g}$ in $\mathrm{Mat}_n(\mathbb{R})$ (see Appendix III.1.1).

The map log unlike its inverse, the exponential map exp, is well defined only in some neighborhood of the unit element. But for the moment we leave aside this relatively small inconvenience.

The more important fact is that the group law is not linearized by the logarithm map. In exponential coordinates it is given by the **Campbell-Hausdorff formula:**[6]

$$(5)\qquad \begin{aligned}&\log(\exp X \cdot \exp Y)\\ &\quad = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) + \cdots .\end{aligned}$$

Of course, it could not be otherwise bearing in mind that the group law is non-commutative. But Dedekind had already pointed out the remedy more than a century ago. He observed that for non-commutative groups the elements g_1g_2 and g_2g_1 can be different but always belong to the same conjugacy class. We can write it as a principle:

On the level of conjugacy classes, the group law is always commutative.

This is a crucial fact in the representation theory of finite groups and is also a foundation of the Cartan–Gelfand–Godement–Harish-Chandra theory of spherical functions.

A new aspect of this phenomenon was observed rather recently in [**DW**] (see Chapter 5, Section 3.5). It turns out that using the logarithm map one can intertwine two kinds of convolution operations: the group convolution of class functions on G and the Lie algebra convolution of $\mathrm{Ad}(G)$-invariant functions on $\mathfrak{g}$. In fact, the precise formulations are given in [**DW**] only for compact and nilpotent Lie groups (the latter can be apparently extended to the case of exponential Lie groups).

In both cases the statements are essentially equivalent to the integral formula for characters (Rule 6 of the User's Guide), although quite another technique is used for the proof in the nilpotent case.

The appearance of coadjoint orbits is now very natural. First, we replace the convolution algebra of class functions on G by the convolution algebra of $\mathrm{Ad}(G)$-invariant functions on $\mathfrak{g}$ and then note that the latter algebra and the ordinary algebra of $K(G)$-invariant functions on $\mathfrak{g}^*$ form the so-called **dual hypergroups**, related by the Fourier transform.

I mention also the other, infinitesimal, way to express the same result. The generalized Fourier transform (see Chapter 4) simplifies not only the

[6]In fact, only the existence of such a formula was proven by E. Campbell and F. Hausdorff and the beautiful (but not very practical) explicit expression for the coefficients was found later by E. B. Dynkin.

convolution but also the infinitesimal characters, i.e. the action of Laplace-Casimir operators. The result can be roughly formulated as follows.

Proposition 3. *All Laplace-Casimir operators become differential operators with constant coefficients (in the canonical coordinates) after conjugating by the operator of multiplication by $p(X)$ and restricting to class functions.*

For semisimple Lie groups a theorem of this sort was first obtained in the thesis of F. A. Berezin in the 1960's.[7]

For the general case the theorem was suggested in my course at Institut Henri Poincaré in 1968 and was later proved by Duflo [**Du3**] and by Ginzburg [**Gi**].

A much more general statement was recently proved by Kontsevich [**Kon**] in the framework of deformation quantization. I would like to quote a sentence from his paper:

Now we can say finally that the orbit method has solid background.

4.2. Physical argument.

This argument is now widely known under the rather lucky name of **geometric quantization**. The idea is to use a correspondence between classical and quantum physical systems.

As we well understand now, there is no canonical and universal correspondence: the quantum world is different from the classical one.

Nevertheless, for many particular systems, the so-called **quantization rules** were formulated. They allow us to construct a quantum system from the classical one. Moreover, the symmetry possessed by a classical system is often inherited by the quantum counterpart.

There are many ways to translate the physical term "quantization" into mathematical language (e.g., there exist algebraic, asymptotic, deformational, path-integral, and other quantizations). All these theories are based on the premise that classical and quantum mechanics are just different realizations of the same abstract scheme.

The goal of geometric quantization is to construct quantum objects from the geometry of classical ones. Historically, there were two origins of this approach:

1) the "quantization rules" of old quantum mechanics, which become more and more elaborate (but still remain adjusted to rather special Hamiltonians defined on special phase spaces);

[7]There were some minor gaps in the Berezin proof noticed by Harish-Chandra during his visit to Moscow in 1966. The appropriate correction appeared the next year.

2) the functor of unitary induction and its generalizations in the representation theory, which allowed us to construct explicitly the unitary duals (or at least a large part of it) for many Lie groups.

It was Bertram Kostant who observed in [**Ko1**] that one can merge these theories into a new one. At the same time Jean-Marie Souriau suggested the same idea in [**S**]. Since then geometric quantization became very popular, especially among physicists.

We have, however, to remark that practically no general results in the non-homogeneous situation were obtained.[8] The quantization rules mentioned above are usually not well defined and sometimes are even contradictory. In the homogeneous situation they are practically equivalent to one or another variant of the induction procedure (see Appendix V.2).

We refer to [**GS1**], [**Ki7**], [**Ko1**], and [**S**] for the accurate definition and basic properties of geometric quantization.

The most interesting and important applications of geometric quantization are related to infinite-dimensional systems. Many of them are only proved "on the physical level". See [**AG**], [**AS**], [**GS1**], [**Vo1**], and [**Wi2**] for further details.

Let us now consider physical systems with a given symmetry group G. We call such a system **elementary** if it cannot be decomposed into smaller parts without breaking the symmetry.

On the classical level the phase space of a physical system with given symmetry group G is a symplectic G-manifold M. For an elementary system this manifold M must be homogeneous.

On the quantum level the phase space of a physical system with given symmetry group G is a projectivization of a Hilbert space $\mathcal{H}$ with a unitary representation of G in $\mathcal{H}$. For an elementary system this representation must be irreducible.

Thus, the quantization principle suggests a correspondence between homogeneous symplectic G-manifolds on the one hand and unirreps of G on the other.

Actually, the situation is slightly more delicate. It is known that the energy function for classical systems is defined up to an additive constant, while for a quantum system the energy is uniquely defined and is usually non-negative.

This shows that the right classical counterpart to quantum systems with the symmetry group G are Poisson G-manifolds rather than symplectic ones. But we have seen in Chapter 1 that homogeneous Poisson G-manifolds are

[8] Except some negative ones. For example, in [**GS1**] it was shown that the general quantization formula does not work even in a low-dimensional non-homogeneous situation.

essentially coadjoint orbits. So we come to the desired correspondence between orbits and representations.

5. Byproducts and relations to other domains

The orbit method stimulated the study of coadjoint orbits and turned out to be related to several other domains that were rapidly developing in the last few decades. We briefly describe here the three most important directions.

5.1. Moment map.

The first general definition of the moment map was given by Souriau [**S**], although its particular cases (e.g., related to the Galileo and Poincaré groups) were known to physicists long ago.

In particular, the famous E. Noether theorem describing the connection between symmetries and invariants in Hamiltonian formalism is simply the moment map for a 1-dimensional Lie group of symmetries.

Most of the new applications of the moment map are related to the notion of **symplectic reduction** (see Appendix II.3.4). This procedure, which also goes back to classical Hamiltonian mechanics, is naturally formulated in terms of the moment map.

Let G be a connected Lie group, (M, σ) a symplectic G-Poisson manifold, and $\mu : M \to \mathfrak{g}^*$ the associated moment map (see Chapter 1). For any coadjoint orbit $\Omega \subset \mathfrak{g}^*$ the set $M_\Omega = \mu^{-1}(\Omega)$ is G-invariant.

Suppose that this set is a smooth manifold and that G acts on it so that all orbits have the same dimension.[9] Then the set $(M_\Omega)_G$ of G-orbits in M_Ω is also a smooth manifold and possesses a canonical symplectic structure.

Indeed, one can easily check that the restriction of σ to $M_F = \mu^{-1}(F)$ is degenerate and $\ker \sigma|_{M_F}$ at a point m coincides with the tangent space to the $Stab\,(F)$-orbit of m. Therefore, σ induces a non-degenerate form on $(M_F)_{Stab\,(F)} \simeq (M_\Omega)_G$.

Exercise 2. Let $M = P^n(\mathbb{C})$ with the symplectic structure induced by the Fubini-Study form. The torus $\mathbb{T}^{n+1}$ that acts on M by the formula

$$(t_0, t_1, \dots, t_n) \cdot (z_0 : z_1 : \cdots : z_n) = (t_0 z_0 : t_1 z_1 : \cdots : t_n z_n).$$

Describe the symplectic reduction of M with respect to the whole torus and with respect to its subgroup $\mathbb{T}^{k+1}, 0 \leq k < n$. ♣

[9]In practical situations these conditions are often violated on a submanifold of lower dimension. Then one has to delete this submanifold or consider so-called orbifolds and more general manifolds with singularities.

This procedure allows us to reduce the study of a mechanical system with the symmetry group G to the study of another system with fewer degrees of freedom (and fewer degrees of symmetry).

Sometimes, it is worthwhile to reverse this procedure and consider a complicated low-dimensional system as a result of the reduction of a simple higher-dimensional system.

We refer to [**AG**] for a survey of the symplectic geometry and its applications and to [**GS2**] for the description of geometric properties of the moment map.

Here we give only the simple algebraic interpretation of the symplectic reduction. In the algebraic approach, the submanifold M_Ω is replaced by the algebra $C^\infty(M_\Omega)$ of smooth functions on this manifold. It is the quotient algebra of $C^\infty(M)$ by the ideal I formed by functions which vanish on M_Ω.

Unfortunately, I is not a Poisson algebra: it is not closed with respect to the bracket operation.

Consider the subalgebra $A \subset C^\infty(M) = \{f \in C^\infty(M) \mid \{f, I\} \subset I\}$. Then I is a Poisson ideal in A, hence A/I inherits a Poisson structure. It turns out that $A/I \simeq C^\infty((M_\Omega)_G)$.

Note that the same manifold $(M_\Omega)_G$ can be obtained as a submanifold of the quotient manifold M_G. In an algebraic approach this means that we take a subalgebra $C^\infty(M)^G \subset C^\infty(M)$ and factorize it over the appropriate ideal J.

5.2. Integrable systems.

This huge domain was intensively developed during the last 30 years. Before that isolated examples were known and no general theory existed. The new era began with the seminal discovery that the so-called Korteweg–de Vries (KdV for short) equation

$$p_t = pp_x + p_{xxx}$$

is a completely integrable system which possesses an infinite series of conservation laws. Since then a lot of important examples of classical and quantum integrable systems were found and several schemes were proposed to explain their appearance (see, e.g., [**DKN**]).

The orbit method is a natural source of homogeneous symplectic manifolds (coadjoint orbits) that can be considered as phase spaces of classical mechanical systems. Note that most of them are not isomorphic to cotangent bundles and therefore do not correspond to a traditional mechanical system. On the other hand, this new kind of phase space includes the following remarkable example.

Example 4. Consider a 2-dimensional sphere S^2 with a symplectic form such that the total volume is an integer $n \geq 1$. As a coadjoint orbit for $\mathfrak{su}(2)$ it admits a quantization via the corresponding unirrep of dimension n. The physical interpretation of this quantum system is a particle with spin $s = \frac{n-1}{2}$, which has no other degree of freedom. So we get a classical counterpart of the notion of spin that for a long time was not believed to exist. ♢

To construct an integrable system, we need either a big family of Poisson commuting functions (in the classical picture) or a big family of commuting operators (in the quantum picture). The so-called **Adler-Kostant scheme** (see, e.g., [**Ko2**] and [**RS**]) provides such a family.

The most simple version of this scheme is based on the decomposition of a Lie algebra $\mathfrak{g}$ into a direct sum of subspaces $\mathfrak{g}_\pm$ that are in fact subalgebras in $\mathfrak{g}$. In this case we can define on $\mathfrak{g}$ the new commutator

$$[X,Y]^{\tilde{}} := [X_+, Y_+] - [X_-, Y_-] \tag{6}$$

where $X_\pm$ denotes the projection of $X \in \mathfrak{g}$ to $\mathfrak{g}_\pm$.

The commutator (6) defines a new Lie algebra structure on $\mathfrak{g}$ and a new Poisson structure on $\mathfrak{g}^*$.

The remarkable fact is that central functions with respect to the first bracket are still commuting in the sense of the second bracket.

Moreover, the Hamiltonian systems corresponding to $H \in P(\mathfrak{g}^*)^G$ admit an explicit description in terms of the **factorization problem**:

$$g = g_+ \cdot g_-, \qquad g \in G,\ g_\pm \in G_\pm. \tag{7}$$

Theorem 1 (Adler–Kostant–Semenov-Tyan-Shanski). a) *All functions in $P(\mathfrak{g}^*)^G$ form a Poisson commuting family with respect to the new structure.*

b) *For a function $H \in P(\mathfrak{g}^*)^G$ define the curves $g_\pm(t)$ in G by the equation*

$$\exp(tdH(F)) = g_+(t)^{-1} g_-(t). \tag{8}$$

Then the trajectory of a point $F \in \mathfrak{g}^*$ under the Hamiltonian flow corresponding to the function H is given by

$$F(t) = K(g_+(t))F = K(g_-(t))F. \tag{9}$$

□

The application of this scheme to different Lie algebras $\mathfrak{g}$ and different points $F \in \mathfrak{g}^*$ gives a uniform construction for most known integrable

systems, including infinite-dimensional systems (such as KdV and its super-analogues, which are related to the Virasoro algebra and its superextensions; see [**OPR**], [**KO**]).

The only deficiency of this approach is that it appeared *post factum*, when almost all interesting examples were discovered by other methods.

6. Some open problems and subjects for meditation

6.1. Functional dimension.

It is well known that:

1) All separable, infinite-dimensional Hilbert spaces are isomorphic.

2) The spaces $C^\infty(M)$ are isomorphic Fréchet spaces for all smooth compact manifolds M of positive dimension.

3) All infinite countable sets are equivalent.

But there is no *natural* isomorphism between

1) $L^2(\mathbb{R}, dx)$ and $L^2(\mathbb{R}^2, dxdy)$.

2) $C^\infty(S^1)$ and $C^\infty(S^2)$.

3) $\mathbb{Z}$ and $\mathbb{Z}^2$.

The non-formal problem is to define the **functional dimension** f-dim of an infinite-dimensional space so that, for example, we have

$$f\text{-dim}\, L^2(\mathbb{R}^n, d^n x) = f\text{-dim}\, C^\infty(\mathbb{R}^n) = n. \tag{10}$$

In order to do this we have to restrict the set of morphisms between our spaces and allow only *natural* morphisms. There are several possible ways to impose such a restriction.

One way is to define some "basic" morphisms and consider as natural only those morphisms that are compositions of basic ones.

For the spaces of smooth functions on manifolds the set of basic morphisms should include

a) multiplication by non-vanishing functions,

b) diffeomorphisms of the underlying manifolds,

c) some integral transformations such as Fourier or Radon transforms.

Another way is to introduce an additional structure on our linear spaces and consider only morphisms that preserve this structure. For example, it is not difficult to show that a compact smooth manifold M is completely determined by the associative algebra $A = C^\infty(M)$ (see Chapter 1) or by the Lie algebra $L = Vect(M)$ of smooth vector fields on M (see below). Indeed, the points of M correspond to maximal ideals in A or to Lie subalgebras of minimal codimension in L.

Here we discuss one more non-formal problem related to the notion of functional dimension:

Show that if $\dim M_1 > \dim M_2$, *then the Lie algebra* $Vect(M_1)$ *is, first, "bigger" and, second, "more non-commutative" than* $Vect(M_2)$.

The answer to one of the possible rigorous versions of the first question was obtained in [**KK**] and [**KKM**]. We describe it below.

Let ξ, η be a pair of vector fields on M. Consider the Lie subalgebra $L(\xi, \eta) \subset Vect(M)$ generated by these fields. It is a bigraded Lie algebra of the form

$$L(\xi, \eta) = FL(x, y)/I(\xi, \eta) \tag{11}$$

where $FL(x, y)$ is a naturally bigraded free Lie algebra with two generators x, y and $I(\xi, \eta) \subset FL(x, y)$ is the kernel of the map $\phi : FL(x, y) \to L(\xi, \eta)$ defined by $\phi(x) = \xi,\ \phi(y) = \eta$.

It turns out that for generic ξ, η the ideal $I(\xi, \eta)$ depends only on $\dim M$. So, for each $n \in \mathbb{N}$ we get a distinguished bigraded ideal $I_n \subset FL(x, y)$ and we define the bigraded Lie algebra $L(n) := FL(x, y)/I_n$.

The growth of the numbers $a_{k,l}(n) := \dim L^{k,l}(n)$ can be considered as characteristic of the size of $Vect(M)$ for n-dimensional M.

Theorem 2 (Conjectured in [**KK**], proved later by Molev).

$$a_{k,l}(1) = p_k(k + l - 1) + p_l(k + l - 1) - p(k + l - 1) \tag{12}$$

where $p(n)$ *is the standard partition function and* $p_k(n)$ *is the number of partitions of* n *into* $\leq k$ *parts (or into parts of size* $\leq k$*).* □

An interesting corollary is that the sequence $a_m(n) = \sum_{k+l=m} a_{k,l}(n)$ for $n = 1$ has **intermediate**, or **subexponential**, growth. Namely,

$$a_m(1) \approx \frac{1}{4\sqrt{3}} e^{\pi\sqrt{\frac{2}{3}m}} \quad \text{which implies} \quad \log\log a_m(1) \approx \frac{1}{2}\log m. \tag{13}$$

A more general result has been obtained by A. I. Molev:

$$\log\log a_m(n) \approx \frac{n}{n+1}\log m. \tag{14}$$

Hence, all Lie algebras of vector fields on smooth manifolds have intermediate growth. In particular, contrary to a widespread delusion, they never contain a free Lie subalgebra that has exponential growth:

$$\sum_{k+l=m} \dim FL^{k,l}(x, y) = \frac{1}{m}\sum_{d|m} \mu(d) 2^{m/d} \approx \frac{2^m}{m}.$$

Many interesting problems arise in connection with the second question, i.e. with the algebraic structure of the algebras $L(n)$ and ideals $I(n)$. We mention only the following.

Conjecture. *The ideal $I(1)$ is spanned by expressions*

$$(15)\qquad \sum_{s\in S_4} \operatorname{sgn} s \cdot \operatorname{ad}(x_{s(1)})\operatorname{ad}(x_{s(2)})\operatorname{ad}(x_{s(3)})\operatorname{ad}(x_{s(4)})x_0, \; x_i \in FL(x, y).$$

There is some evidence in favor of this conjecture. In particular, it seems that $L(1)$ admits a faithful representation as the Lie algebra of vector fields on an infinite-dimensional manifold, tangent to a 1-dimensional foliation.

In conclusion we repeat the main question:

Give a rigorous definition of the functional dimension of a unirrep so that Rule 9 *of the User's Guide holds.*

6.2. Infinitesimal characters.

The general proof of the modified Rule 7 was obtained independently in [**Du3**] and [**Gi**]. The proof is rather involved and analytic in nature. But the statement itself is purely algebraic and certainly can be proved algebraically.

My own attempt to do it was broken by the discovery (cf. Appendix III.2.3) that the manifold A_n of structure constants of n-dimensional Lie groups is highly reducible. So, it is doubtful that one can prove the statement just using the defining equations of A_n.

Another approach was suggested in [**KV**] but the problem is still open.

Quite recently I learned that M. Kontsevich has found a new proof based on the computation of a functional integral suggested by a variant of quantum field theory.

This is probably the right solution to the problem (cf. the quotation in the end of section 4.1). In very general terms, the advantage of this new approach is that one considers general Poisson manifolds (and not only those which are related to Lie algebras). So, the situation is similar to the famous Atiyah-Singer Index Theorem for differential operators, which was proved only after passing to the more general context of pseudo-differential operators.

6.3. Multiplicities and geometry.

Let G be a compact simply connected Lie group, let T be its maximal connected abelian subgroup, and let $\mathfrak{g} \supset \mathfrak{t}$ be the corresponding Lie algebras that are identified with their duals via an $\operatorname{Ad}(G)$-invariant scalar product. Let $P \subset i\mathfrak{t}^*$ be the weight lattice, $Q \subset P$ a root sublattice, $\rho \in P$ the sum of the fundamental weights, and $\Omega_\lambda \subset \mathfrak{g}^*$ the coadjoint orbit of the point $i\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$. We denote by p the natural projection of $\mathfrak{g}^*$ on $\mathfrak{t}^*$.

It is known that for any $\lambda \in P_+$ the set $C_\lambda = p(\Omega_{\lambda+\rho})$ is the convex hull of $|W|$ different points $\{iw(\lambda + \rho), \; w \in W\}$.

Let us call an **elementary cell** in $\mathfrak{t}^*$ the set $C_0 = p(\Omega_\rho)$, as well as all its translations by elements of P. One can check that C_λ is the union of elementary cells centered at the points of $(\lambda + Q) \cap p(\Omega_\lambda)$. For more precise results see Chapter 5.

A difficult and non-formal question is:

How is the multiplicity $m_\lambda(\mu)$ of a weight μ in the unirrep π_λ related to the geometry of the sets $p^{-1}(C_0 + \mu)$ and $p^{-1}(\mu)$?

6.4. Complementary series.

The orbit method apparently leaves no place for the complementary series of representations of semisimple groups. Indeed, according to the ideology of the orbit method, the partition of $\mathfrak{g}^*$ into coadjoint orbits corresponds to the decomposition of the regular representation into irreducible components. But unirreps of the complementary series by definition do not contribute to this decomposition.

One possible solution to this paradox is based on a remark made in one of the early papers by Gelfand–Naimark. They observed that for non-compact semisimple groups there is a big difference between $L^1(G, dg)$ and $L^2(G, dg)$ due to the exponential growth of the density of the Haar measure.

The effect of this difference can be explained as follows. One of the ingredients of the orbit method is the generalized Fourier transform (see Chapter 4) from the space of functions on G to the space of functions on $\mathfrak{g}^*$, which is the composition of two maps:

1. the map from functions on G to functions on $\mathfrak{g}$:

$$f \longmapsto \phi \quad \text{where } \phi(X) = \sqrt{\frac{d(\exp X)}{dX}} \cdot f(\exp X);$$

2. the usual Fourier transform that sends functions on $\mathfrak{g}$ to functions on $\mathfrak{g}^*$.

The image of $L^2(G, dg)$ under the generalized Fourier transform consists of square integrable functions (at least if we consider functions with the support in the domain $\mathcal{E}$ where the exponential map is one-to-one). But the image of $L^1(G, dg)$ consists of much nicer functions that admit an analytic continuation from $\mathfrak{g}^*$ to some strip in $\mathfrak{g}^*_{\mathbb{C}}$.

So, one can try to associate complementary series of unirreps with those G-orbits that lie inside this strip and are invariant under complex conjugation. One can check that in the simplest case $G = SL(2, \mathbb{R})$ this approach leads to the correct integral formula for the generalized character of a representation of the complementary series.

I believe that the problem deserves further investigation.

6.5. Finite groups.

The orbit method can be developed in a very interesting direction. Namely, one can try to apply it to some finite groups of the kind $G(\mathbb{F}_q)$ where G is an algebraic group defined over $\mathbb{Z}$ and $\mathbb{F}_q$ is a finite field of q elements. For instance, we can consider the sequence G_n of classical groups: $GL(n), SO(n), Sp(n)$.

We are mainly interested in the asymptotic properties of harmonic analysis on $G_n(\mathbb{F}_q)$ when q is fixed and n goes to infinity. In particular I want to advertise here some principal questions:

1. What is the asymptotics of the number of coadjoint orbits for $G_n(\mathbb{F}_q)$?

2. Can one describe the "generic" or "typical" coadjoint orbit?

3. More generally, which characteristics of orbits and representations can one deal with for the groups of "very large matrices", say of order 10^{10} (or even 20) over a finite field?

(Note that the simplest numerical questions about these groups are out of the range of modern computers.)

Of course, these questions make sense not only for triangular groups and their analogs (unipotent radicals of classical groups). For instance, one can try to find the answer for $GL(n,\ \mathbb{F}_q)$, using the results from [**Ze**].

6.6. Infinite-dimensional groups.

This is perhaps the most promising generalization of the orbit method. In the situation when there is no general theory and many very important and deep examples, the empirical value of the orbit method is hard to overestimate.

The most intriguing and important question is related to the Virasoro-Bott group. It can be formulated as follows:

How one can explain the rather complicated structure of discrete series of unirreps of Vir in terms of the rather simple structure of the set of coadjoint orbits?

Note that the very interesting paper [**AS**] gives an answer to this question, although it is written on the physical level of accuracy and needs a translation into mathematical language.

Following the example of D. Knuth, I conclude the section by the following general

Exercise 3.* Formulate and solve other problems concerning the application of the orbit method to infinite-dimensional groups. ♣

Appendix I

Abstract Nonsense

In this appendix we collect some definitions and results from topology, category theory, and homological algebra. They are formulated in a very abstract form and at first sight seem to claim nothing about everything. That is why homological algebra has the nickname "abstract nonsense".

I hope, however, that the examples in the text will convince you that this material is really useful in many applications and, in particular, in representation theory.

Note that in the examples we use some notions and results described elsewhere. So, the reader must consult the index and look for the information in the main text or in the other appendices.

1. Topology

1.1. Topological spaces.

The notion of a topological space is useful, but not strictly necessary for the understanding of the main part of the book. We briefly discuss this notion here to make the reader more comfortable.

Roughly speaking, a **topological space** is a set X where the notion of a neighborhood of a point is defined. In the ordinary Euclidean space $\mathbb{R}^n$ the role of neighborhoods are played by the open balls.

It is well known that all basic concepts of analysis can be formally defined using only the notion of a neighborhood:

interior point: $x \in X$ is an interior point of $S \subset X$ if it belongs to S together with some neighborhood;

exterior point: $x \in X$ is an exterior point of $S \subset X$ if it is an interior point of the complement $\neg S = X \backslash S$;

boundary point: $x \in X$ is a boundary point of $S \subset X$ if it is neither interior nor exterior for S;

open set: a subset $S \subset X$ is open if all of its points are interior;

closed set: a subset $S \subset X$ is closed if it contains all its boundary points;

connected space: a topological space X is connected if it cannot be represented as a union of two disjoint non-empty open subsets;

limit of the sequence $\{x_n\} \subset X$: a point $a \in X$, such that every neighborhood of it contains all members of the sequence except, maybe, finitely many of them;

continuous map: a map $f : X \longrightarrow Y$ is continuous if for any open subset $S \subset Y$ its preimage $f^{-1}(S)$ is open in X;

sequentially continuous map: a map $f : X \longrightarrow Y$ is sequentially continuous if it commutes with limits:

$$f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) \qquad \text{for any convergent sequence } \{x_n\} \text{ in } X.$$

The primary object of abstract topology is the notion of an open set. From this we can derive the definition of a **neighborhood** of $a \in X$ as any open subset $O \subset X$ that contains a.

So, to define a topology on a set X we have only to specify the collection $\mathcal{O}$ of open subsets in X. It turns out that $\mathcal{O}$ can be chosen arbitrarily subject to three very simple axioms:

1. The union of any family of open sets is open.
2. The intersection of a finite family of open sets is open.
3. The empty set $\emptyset$ and the whole space X are open.

It is a miracle how from this simple set of axioms one can derive deep and important consequences, such as the existence and/or uniqueness of solutions to very complicated systems of algebraic and differential equations.[1]

1.2. Metric spaces and metrizable topological spaces.

Of course, the most interesting topological spaces are those which naturally appear in different domains of mathematics and its applications. For many of them the topology can be defined in a special way. Namely, assume that the space X in question admits a natural notion of the **distance** $d(x, y)$ between the points x and y. It is supposed that the distance satisfies the following three conditions:

[1] As a byproduct we also get a lot of papers of minor interest, but this is not a miracle.

1. Positivity: $d(x, y) > 0$ for $x \neq y$ and $d(x, y) = 0$ for $x = y$.
2. Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. Triangle Axiom: $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The space X with a distance $d : X \times X \longrightarrow \mathbb{R}_+$ satisfying the above conditions is called a **metric space**. It is usually denoted by (X, d).

In a metric space (X, d) we can define for any $\epsilon > 0$ the ϵ-**neighborhood** of a point a as the set

$$U_\epsilon(a) = \{x \in X \mid d(x, a) < \epsilon\}.$$

The set $U_\epsilon(a)$ is also called an **open ball** with center a and radius ϵ.

So, every metric space (X, d) can be endowed with a topology. Namely, the subset $A \subset X$ is called open if it is a union of an arbitrary family of open balls. Topological spaces of this kind are called **metrizable**. Most topological spaces used in our book are metrizable, although there are a few exceptions.

Exercise 1. a) Prove that for any topological space a continuous map is sequentially continuous.

b) Show that for metrizable spaces the converse is also true: any sequentially continuous map is continuous. ♣

Among all metric spaces we can distinguish a very important class of complete metric spaces defined as follows. Call a sequence $\{x_n\}$ in a metric space (X, d) a **fundamental sequence** (or **Cauchy sequence**) if

$$\lim_{m,n \to \infty} d(x_m, x_n) \to 0. \tag{1}$$

More minutely: for any $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that

$$d(x_m, x_n) < \epsilon \quad \text{for all } m > N, n > N.$$

A metric space (X, d) is called **complete** if every fundamental sequence converges, i.e. has a limit in X.

Two useful properties of complete metric spaces are:

Theorem on nested balls. *Assume that a sequence of closed balls*

$$B_n = \{x \in X \mid d(x, a_n) \leq r_n\}$$

has the properties: a) $B_{n+1} \subset B_n$ *for all* n, b) $r_n \to 0$ *when* $n \to \infty$. *Then the intersection* $\bigcap_n B_n$ *contains exactly one point.* □

Theorem on contracting maps. *Assume that for some* $\lambda \in (0,1)$ *a map* $f : X \longrightarrow X$ *has the property*

$$d(f(x),\, f(y)) \leq \lambda \cdot d(x,\, y). \tag{2}$$

Then f *has a unique fixed point* $x_0 \in X$ *(i.e. such that* $f(x_0) = x_0$*).* □

Not all metric spaces are complete, but every metric space $(X,\, d)$ has a so-called **completion** $(\widetilde{X},\, \widetilde{d})$. By definition, $(\widetilde{X},\, \widetilde{d})$ is a complete metric space such that $\widetilde{X}$ contains a dense subset X_0 which is isometric to X.

Informally speaking, $\widetilde{X}$ is obtained from X by adding some "ideal" points, which are limits in $\widetilde{X}$ of those Cauchy sequences that have no limits in X.

Example 1. The real line $\mathbb{R}$ with the standard distance $d(x,\, y) = |x - y|$ is the completion of the set $\mathbb{Q}$ of rational numbers.

Here the ideal points are irrational numbers. The infinite decimal representation of x:

$$x = x_n \cdots x_1 x_0, x_{-1} \cdots x_{-k} \cdots = \sum_{-\infty}^{j=n} x_j \cdot 10^j$$

provides a Cauchy sequence of rational numbers converging to x.

One can define other distances on $\mathbb{Q}$ which are translation-invariant and in a way compatible with multiplication. Namely, for any prime number p the so-called ***p*-adic norm** is defined as $\|r\|_p = p^{-k}$ if the rational number r has the form $r = p^k \cdot \frac{m}{n}$ where m and n are coprime with p. We define the p-adic distance by

$$d_p(r_1,\, r_2) = \|r_1 - r_2\|_p. \tag{3}$$

It turns out that the completion of the metric space $(\mathbb{Q},\, d_p)$ is a field $\mathbb{Q}_p$ where all arithmetic operations are extended by continuity from $\mathbb{Q}$. The elements of $\mathbb{Q}_p$ are called ***p*-adic numbers**; they can be written as a semi-infinite string of digits $\{0,\, 1,\, \ldots,\, p-1\}$:

$$a = \ldots a_n \ldots a_1 a_0, a_{-1} \ldots a_{-k} = \sum_{j=-k}^{+\infty} a_j \cdot p^j. \tag{4}$$

Note that the series is convergent in $(\mathbb{Q}_p,\, d_p)$.

It is worthwhile to mention that the closure $\mathbb{Z}_p$ of the subset $\mathbb{Z} \subset \mathbb{Q}$ coincides with the closure of the subset $\mathbb{N} \subset \mathbb{Z}$ and is compact and homeomorphic to the Cantor set. The elements of $\mathbb{Z}_p$ are called ***p*-adic integers**. In the notation of (4) they are characterized by the property $k = 0$. ♢

Exercise 2. Show that for any $a \in \mathbb{Z}_p$ the limit $\lim_{n\to\infty} a^{p^n}$ exists. It is called the p-adic signum of a and is denoted by $\mathrm{sign}_p(a)$. Prove the following properties of this function:

a) $\mathrm{sign}_p(ab) = \mathrm{sign}_p(a)\mathrm{sign}_p(b)$;

b) the number $s = \mathrm{sign}_p(a)$ satisfies the equation $s^p = s$, hence is either 0 or a $(p-1)$-st root of unity;

c) the value of $\mathrm{sign}_p(a)$ depends only on the last digit a_0 in the notation of (4). ♣

Example 2. The Hilbert space $L^2(\mathbb{R}, dx)$ is the completion of each of the following spaces:

a) the space $\mathcal{A}(\mathbb{R})$ of smooth functions f with compact support;

b) the space $C_0(\mathbb{R})$ of continuous functions f with compact support;

c) the space of step-functions f with compact support;

d) the space of functions $f(x) = P(x)e^{-x^2}$ where P is a polynomial in x.

All are endowed with the distance

$$d(f_1, f_2) = \sqrt{\int_{\mathbb{R}} |f_1(x) - f_2(x)|^2 dx}.$$

Here the ideal points are Lebesgue measurable functions on $\mathbb{R}$ with an integrable square that do not belong to the subspaces listed above. In most applications of the Hilbert space $L^2(\mathbb{R}, dx)$ these elements play only a subsidiary role to make the whole space complete. ♢

2. Language of categories

2.1. Introduction to categories.

Since the beginning of the 20-th century all mathematical theories have been founded on the set-theoretic base. This means that every object of study is defined as a set X with some additional structures. The notion of a structure itself is also defined in the set-theoretic manner as a point in some auxiliary set constructed from X.

Example 3. A map $\phi : A \longrightarrow B$ can be defined in set-theoretic terms as a subset $\Gamma_\phi \subset A \times B$, the **graph** of ϕ, such that the projection $\Gamma_\phi \longrightarrow A$ is a bijection;

a group is a set G with a distinguished point in $G^{(G\times G)}$ which specifies the multiplication law;

a topological space is a set X with a special point in 2^{2^X} describing the collection of open subsets in X;

a vector space over a field K is a set V with a point in $V^{(V \sqcup K) \times V}$, which determines the addition law and the multiplication of vectors by numbers; etc.

(In this example we used the following, now practically standard, notation: B^A denotes the set of all maps from A to B and 2^X denotes the set of all subsets of X that is in a natural bijection with the set of all maps from X to a two-point set.) $\diamondsuit$

There exists, however, a quite different approach to the foundation of mathematics: one of a more sociological[2] nature. The point is to consider mathematical objects as members of a certain commonwealth and characterize them by their relations to other objects of the same nature. The basic notion of this new approach is the notion of a category.

To define a **category** $\mathcal{C}$ we have

a) to specify a family (not necessarily a set!) $Ob\mathcal{C}$ of **objects** of $\mathcal{C}$;

b) for any pair of objects $X, Y \in Ob\mathcal{C}$ to define a set of **morphisms** from X to Y, denoted $\mathrm{Mor}_{\mathcal{C}}(X, Y)$;

c) for each triple $X, Y, Z \in Ob\mathcal{C}$ to define a map

$$\mathrm{Mor}_{\mathcal{C}}(X, Y) \times \mathrm{Mor}_{\mathcal{C}}(Y, Z) \to \mathrm{Mor}_{\mathcal{C}}(X, Z): \quad (f, g) \mapsto g \circ f,$$

which is called the **composition** of morphisms.

It is assumed that

1. The composition law is associative: $(f \circ g) \circ h = f \circ (g \circ h)$ whenever defined.

2. For each object X there exists a unique **unit morphism** $1_X \in \mathrm{Mor}_{\mathcal{C}}(X, X)$, which plays the role of left and right unit:

$$f \circ 1_X = 1_Y \circ f = f \quad \text{for any } f \in \mathrm{Mor}_{\mathcal{C}}(X, Y).$$

For many important categories their objects are sets (usually with some additional structures), morphisms are maps (preserving the additional structures), composition is the usual composition of maps, and 1_X is the identity map Id. In other words, they are **subcategories** of the category $\mathcal{S}ets$.

In particular, this is true for the following categories:

$\mathcal{S}ets$: objects are sets, morphisms are maps, so that $\mathrm{Mor}_{\mathcal{S}ets}(A, B) = B^A$;

$\mathcal{V}ect_K$: objects are vector spaces over a given field K, morphisms are K-linear operators, so that $\mathrm{Mor}_{\mathcal{V}ect_K}(V, W) = \mathrm{Hom}_K(V, W)$;

[2] I borrowed this epithet from Yu. I. Manin.

$\mathcal{G}r$: objects are groups, morphisms are group homomorphisms;

$\mathcal{M}an$: objects are smooth manifolds, morphisms are smooth maps;

$\mathcal{LG}$: objects are Lie groups, morphisms are smooth homomorphisms.

But there are categories of a quite different kind and we consider some useful examples below.

Note that the family of all categories[3] also forms a sort of category

$\mathcal{C}at$: objects are categories and morphisms are the so-called functors which we now describe.

To define a **functor** F from a category $\mathcal{C}_1$ to another category $\mathcal{C}_2$ we have to specify:

a) an object $F(X) \in Ob\,\mathcal{C}_2$ for any $X \in Ob\,\mathcal{C}_1$ and

b) a morphism $F(\phi) \in \mathrm{Mor}_{\mathcal{C}_2}(F(X), F(Y))$ for any $\phi \in \mathrm{Mor}_{\mathcal{C}_1}(X, Y)$ so that

$$F(1_X) = 1_{F(X)}, \qquad F(\phi \circ \psi) = F(\phi) \circ F(\psi) \quad \text{whenever it makes sense.}$$

For any category $\mathcal{C}$ one can define the **dual category** $\mathcal{C}^\circ$. It has the same class of objects as $\mathcal{C}$. A morphism from A to B in $\mathcal{C}^\circ$ is by definition a morphism from B to A in $\mathcal{C}$. The composition $f \circ g$ in $\mathcal{C}^\circ$ is defined as the composition $g \circ f$ in $\mathcal{C}$.

Example 4. There exists a natural duality functor $^* : \mathcal{V}ect_K \to (\mathcal{V}ect_K)^\circ$ which sends a space V to the dual space $V^* := \mathrm{Hom}_K(V, K)$ and a linear operator $A : V \to W$ to the adjoint operator $A^* : W^* \to V^*$. In Appendix VI this functor is extended to the category of Banach spaces and (in a different way) to the category of Hilbert spaces. ♢

Often a category is represented graphically: objects are denoted by points or small circles and morphisms by arrows. The transfer to the dual category means just "reversing the arrows".

A diagram constructed from objects and morphisms of some category is called **commutative** if the following condition is satisfied:

The composition of arrows along any path joining two objects X and Y does not depend on the choice of the path.

Example 5. The commutative diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{A} & V_1 \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q} \\ V_2 & \xrightarrow[B]{} & V_2 \end{array}$$

[3]More precisely, here we have to restrict ourselves by the so-called **small categories**, which we do not define here since it is not essential for understanding the material.

in $\mathcal{V}ect_K$ means the relation $QA = BQ$ between the linear operators $A: V_1 \longrightarrow V_1$, $B: V_2 \longrightarrow V_2$, and $Q: V_1 \longrightarrow V_2$. ♢

2.2. The use of categories.

The categorical approach allows us to unify many definitions and constructions that are used independently in different domains of mathematics.

A simple example: the equivalence of sets, homeomorphism of topological spaces, isomorphism of groups or algebras, and diffeomorphism of smooth manifolds are all particular cases of the general notion of an isomorphism of objects in a category, which is defined as follows.

Two objects $X, Y \in Ob\mathcal{C}$ are called **isomorphic** if there exist morphisms $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y), g \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ such that $f \circ g = 1_Y$, $g \circ f = 1_X$.

Another useful categorical notion is the notion of a universal object. An object $X \in Ob\mathcal{C}$ is called a **universal** (or **initial**) object if for any $Y \in Ob\mathcal{C}$ the set $\mathrm{Mor}_{\mathcal{C}}(X, Y)$ contains exactly one element. (Graphically: there is exactly one arrow from the point X to any point Y.)

The dual notion of a **couniversal** (or **final**) object is obtained by reversing the arrows: Y is final if there is exactly one morphism from any object X to Y.

Note that any two universal objects X, X' in a given category $\mathcal{C}$ are canonically isomorphic. Indeed, by definition there exist unique morphisms $\alpha : X' \longrightarrow X$ and $\beta : X \longrightarrow X'$. The composition $\alpha \circ \beta$ is a morphism from X to itself which is forced to be 1_X since X is universal. The same argument implies $\beta \circ \alpha = 1_{X'}$.

As an application we define here the direct sum and direct product of two objects X_1 and X_2 in a category $\mathcal{C}$. To do this, we construct an auxiliary category $\mathcal{C}(X_1, X_2)$. The objects of $\mathcal{C}(X_1, X_2)$ are triples (a_1, a_2, Y) where Y can be any object of $\mathcal{C}$, $a_1 \in \mathrm{Mor}_{\mathcal{C}}(X_1, Y)$, and $a_2 \in \mathrm{Mor}_{\mathcal{C}}(X_2, Y)$.

Graphically, an object of $\mathcal{C}(X_1, X_2)$ looks as in the following diagram:

$$X_1 \xrightarrow{a_1} Y \xleftarrow{a_2} X_2.$$

A morphism from the object (a_1, a_2, Y) to the object (b_1, b_2, Z) is defined as a morphism ϕ from Y to Z such that the following diagram is commutative:

$$\begin{tikzcd}
X_1 \arrow[r, "a_1"] \arrow[d, "1_{X_1}"'] & Y \arrow[d, "\phi"'] & X_2 \arrow[l, "a_2"'] \arrow[d, "1_{X_2}"] \\
X_1 \arrow[r, "b_1"'] & Z & X_2 \arrow[l, "b_2"]
\end{tikzcd}$$

If the category $\mathcal{C}(X_1, X_2)$ has a universal object (i_1, i_2, X) (recall that all such objects are isomorphic), then X is called the **direct sum** of X_1 and X_2

in $\mathcal{C}$ and morphisms i_1, i_2 are called **canonical embeddings** of summands into the sum.

Exercise 3. Show that direct sums exist for all couples of objects in the following categories and describe them.

a) $\mathcal{S}ets$, b) $\mathcal{V}ect_K$, c) $\mathcal{G}r$, d) $\mathcal{M}an$.

Answer: a) and d) the disjoint union, b) the direct sum of vector spaces, c) the free product of groups. ♣

The definition of the **direct product** in $\mathcal{C}$ is obtained from the above definition by reversing the arrows. In other words, the direct product is the direct sum in the dual category $\mathcal{C}^\circ$. It looks as in the diagram $X_1 \xleftarrow{p_1} X \xrightarrow{p_2} X_2$. So, the direct product X always comes together with **canonical projections** of a product to the factors.

Exercise 4. Describe the operation of direct product for the categories listed above.

Answer: a) and d) the direct (Cartesian) product, b) the direct sum of vector spaces, c) the direct product of groups. ♣

Exercise 5. Extend the definitions of direct sum and direct product to an arbitrary family $\{X_\alpha\}_{\alpha\in A}$ of objects.

Hint. Use an auxiliary category $\widetilde{\mathcal{C}}$ whose objects have the form $(X, \{f_\alpha\})$, where $X \in \mathcal{O}b\mathcal{C}$ and $f_\alpha \in \mathrm{Mor}_{\mathcal{C}}(X_\alpha, X)$ for the direct sum, whereas $f_\alpha \in \mathrm{Mor}_{\mathcal{C}}(X, X_\alpha)$ for the direct product. ♣

Let $\mathcal{C}$ be any category and $X \in \mathcal{O}b\ \mathcal{C}$. The correspondence $A \mapsto \mathrm{Mor}_{\mathcal{C}}(X, A)$ can be extended to a functor $F_X : \mathcal{C} \to \mathcal{S}ets$. Namely, for $\phi \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ we define $F_X(\phi) : \mathrm{Mor}_{\mathcal{C}}(X, A) \to \mathrm{Mor}_{\mathcal{C}}(X, B)$ by $F_X(\alpha) = \phi \circ \alpha$. Often the object X can be uniquely reconstructed from this functor F_X.[4]

When a functor $F : \mathcal{C} \to \mathcal{S}ets$ has the form $F = F_X$, it is called a **representable functor** and the object X is called **the representing object**. Even when this is not the case, it is convenient to consider non-representable functors from $\mathcal{C}$ to $\mathcal{S}ets$ as "generalized objects" of $\mathcal{C}$.

For a more detailed introduction to the categorical ideology we refer to [**Gr**] and [**M**].

2.3.* Application: Homotopy groups.

Topology plays an essential role in the representation theory of Lie groups. In particular, low-dimensional homotopy groups are used. We assume that the reader is acquainted with the notion of the fundamental

[4]The analog of this statement in ordinary life is the principle: "tell me who your friends are and I will tell you who you are".

group π_1. The quickest way to introduce the higher homotopy groups π_k is via category theory.

Consider the category $\mathcal{HT}$ whose objects are pointed topological spaces (i.e. topological spaces with a marked point) and morphisms are homotopy classes of continuous maps which send a marked point to a marked point.

Recall that two continuous maps $f_0,\, f_1 : (X,\, x) \longrightarrow (Y,\, y)$ belong to the same **homotopy class** $[f]$ if there exists a continuous map $F : X \times [0,1] \longrightarrow Y$ such that $F|_{X\times\{0\}} = f_0,\ F|_{X\times\{1\}} = f_1$ and $F|_{x\times[0,1]} \equiv y$. Two topological spaces have the same **homotopy type** if they are equivalent objects of $\mathcal{HT}$.

Sometimes it is convenient to consider these homotopy types as true objects of $\mathcal{HT}$.

Let $(S^n,\, s)$ be the standard unit sphere in $\mathbb{R}^{n+1}$ given by the equation $(x^0)^2 + \cdots + (x^n)^2 = 1$ with the marked point $x^0 = 1,\ x^k = 0$ for $k \neq 0$.

The collection of sets

$$\pi_n(X,\, x) := \mathrm{Mor}_{\mathcal{HT}}\big((S^n,\, s),\, (X,\, x)\big), \quad n \geq 0, \tag{5}$$

has a very rich algebraic structure. It is the main object of **homotopy theory**.

In particular, for $n \geq 1$ every set $\pi_n(X)$ is endowed with a group law, which comes from the natural morphism $S^n \longrightarrow S^n + S^n$ (direct sum in the category $\mathcal{HT}$). Therefore, $\pi_n(X)$ is called the n-th **homotopy group** of X.

It is known that the group is abelian for $n \geq 2$. Moreover, if X itself is a topological group, then all $\pi_n(X),\ n \geq 0$, have a group structure, which is abelian for $n \geq 1$.

A very useful tool is the so-called exact sequence of a fiber bundle.

Proposition 1. *For any fiber bundle $F \xrightarrow{i} E \xrightarrow{p} B$ there is an exact sequence of homotopy groups:*

$$\cdots \xrightarrow{\delta} \pi_k(F) \xrightarrow{i_*} \pi_k(E) \xrightarrow{p_*} \pi_k(B) \xrightarrow{\delta} \pi_{k-1}(F) \xrightarrow{i_*} \cdots . \tag{6}$$

3. Cohomology

3.1. Generalities.

The general idea of homology and cohomology is very simple. A sequence of abelian groups and homomorphisms

$$\cdots \longrightarrow C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \longrightarrow \cdots \tag{7}$$

is called a **chain complex** if

(8) $$\partial_k \circ \partial_{k+1} = 0 \quad \text{for all } k.$$

Denote the kernel of ∂_k by Z_k and the image of ∂_{k+1} by B_k.

The elements of C_k are called **k-chains**, the elements of Z_k are called **k-cycles**, and the elements of B_k are called **k-boundaries**.

The relation (8) implies that every boundary is a cycle. The quotient group $H_k = Z_k/B_k$ is called the **k-th homology group** of the complex (7). The image of a cycle $z \in Z_k$ in H_k is usually denoted by $[z]$ and is called the **homology class** of z.

Sometimes it is convenient to consider the direct sum $C = \bigoplus_k C_k$ and replace the family $\{\partial_k\}$ by the single **boundary operator** $\partial : C \longrightarrow C$ such that $\partial|_{C_k} = \partial_k$. Then (8) takes the simple form $\partial^2 = 0$.

Reversing the arrows, we get the definition of **cochains, cocycles, coboundaries**, and **cohomology groups**. Traditionally, the following terminology is used:

C^k – the group of k-cochains
$d_k : \mathbb{C}^k \longrightarrow C^{k+1}$ – coboundary operator
$B^k = \operatorname{im}(d_{k-1})$ – the group of k-coboundaries
$Z^k = \ker(d_k)$ – the group of k-cocycles
$H^k = Z^k/B^k$ – the k-th cohomology group.

This general scheme has many concrete realizations. We describe below only three of them that are especially used in representation theory.

3.2. Group cohomology.

Let G be a group and M a G-module, i.e. an abelian group on which G acts by automorphisms.

Define a **homogeneous k-cochain** as a G-equivariant map

$$\widetilde{c} : G^{k+1} = \underbrace{G \times G \times \cdots \times G}_{k+1 \text{ times}} \longrightarrow M$$

where G acts on G^{k+1} by the left shift: $g \cdot (g_0, \ldots, g_k) = (gg_0, \ldots, gg_k)$.

In other words, a homogeneous k-cochain $\widetilde{c}$ is an M-valued function on G^{k+1} with the property

(9) $$\widetilde{c}(gg_0, \ldots, gg_k) = g \cdot \widetilde{c}(g_0, \ldots, g_k).$$

The set $\widetilde{C}^k(G, M)$ of all G-equivariant maps is a group under the usual addition of functions.

Define the differential, or coboundary operator

$$\widetilde{d}: \widetilde{C}^k(G, M) \longrightarrow \widetilde{C}^{k+1}(G, M)$$

by the formula

$$(\widetilde{d}\widetilde{c})(g_0, \dots, g_{k+1}) = \sum_{i=0}^{k+1} (-1)^i \widetilde{c}(g_0, \dots, \widehat{g_i}, \dots, g_{k+1}) \tag{10}$$

where the hat sign $\widehat{\ }$ means that the corresponding argument is omitted.

It is clear that $\widetilde{d}^2 = \widetilde{d} \circ \widetilde{d} = 0$. Indeed, the term

$$\widetilde{c}(g_0, \dots, \widehat{g_i}, \dots, \widehat{g_j}, \dots, g_{k+1})$$

enters in $\widetilde{d}^2\widetilde{c}$ twice with opposite signs.

Technically, it is more convenient to deal with cochains written in non-homogeneous form. Note that the condition (9) implies that the element $\widetilde{c} \in C^k(G, M)$ is completely determined by the values $c(g_1, \dots, g_k) := \widetilde{c}(e, g_1, g_1g_2, \dots, g_1 \cdots g_k)$. Namely, we have

$$\widetilde{c}(g_0, g_1, \dots, g_k) = g_0 \cdot c(g_0^{-1}g_1, g_1^{-1}g_2, \dots, g_{k-1}^{-1}g_k).$$

We call the quantity c the **non-homogeneous** form of $\widetilde{c}$. It is just an arbitrary M-valued function on G^k. The coboundary operator in the non-homogeneous form looks like

$$\begin{aligned} dc\,(g_1, \dots, g_{k+1}) &= g_1 \cdot c(g_2, \dots, g_{k+1}) \\ &+ \sum_{i=1}^{k} (-1)^i\, c(g_1, \dots, g_ig_{i+1}, \dots, g_{k+1}) + (-1)^{k+1} c(g_1, \dots, g_k). \end{aligned} \tag{11}$$

This formula looks less natural than (10) but in return non-homogeneous cochains have one argument less. Later on we shall always use the non-homogeneous form for cochains.

Let us study the cases of small k in more detail.

The case $k = 0$. Here $C^0(G, M) \cong M$ and the differential is $(dm)(g) = g \cdot m - m$. So, $H^0(G, M) = Z^0(G, M) = M^G$, the set of G-invariant elements of M.

The case $k = 1$. Here $dc(g_1, g_2) = g_1 \cdot c(g_2) - c(g_1g_2) + c(g_1)$. So, a 1-cocycle is a map $c: G \longrightarrow M$ satisfying the **cocycle equation**:

$$c(g_1g_2) = c(g_1) + g_1 \cdot c(g_2). \tag{12}$$

We shall see several appearances of this equation in the main part of the book.

The cocycle c is trivial (i.e. $[c] = 0$) if it has the form

$$c(g) = g \cdot m - m \quad \text{for some } m \in M.$$

The trivial cocycles form a coboundary group isomorphic to M/M^G.

In the special case when M is a trivial G-module, the coboundary group reduces to $\{0\}$ and $H^1(G, M) = Z^1(G, M)$ coincides with the group $\mathrm{Hom}(G, M)$ of all homomorphisms of G to M.

The case $k = 2$. Here we have

$$dc(g_1, g_2, g_3) = g_1 \cdot c(g_2, g_3) - c(g_1g_2, g_3) + c(g_1, g_2g_3) - c(g_2, g_3).$$

The trivial cocycles have the form

$$c(g_1, g_2) = g_1 \cdot b(g_2) - b(g_1g_2) + b(g_1) \quad \text{for some } b: G \longrightarrow M.$$

We shall see the interesting example of a 2-cocycle in Appendix V when we discuss the Mackey Inducibility Criterion.

Remark 1. There is a metamathematical statement:

All cocycles are essentially trivial.

This means the following: if we suitably extend the collection of cochains, then any cocycle becomes a coboundary.

Another formulation of this principle is due to well-known physicist L. D. Faddeev who used to say:

Cohomology are just the functions but with singularities.

♡

Many important theorems (including those that were discovered before the cohomology era) can be reformulated as a claim that certain cohomology groups are trivial or become trivial after an appropriate extension of the cochain group.

3.3. Lie algebra cohomology.

This and the next sections use a basic knowledge of smooth manifolds and Lie groups. The reader who has doubts about possessing this knowledge must consult Appendices II and III.

Let G be a connected and simply connected Lie group, and let H be a closed connected subgroup. It is well known that all geometric and topological questions about the homogeneous manifold $M = G/H$ can be translated into pure algebraic questions about the Lie algebras $\mathfrak{g} = \mathrm{Lie}\,(G)$ and

$\mathfrak{h} = \mathrm{Lie}\,(H)$. In particular, we can ask about the cohomology $H(M)$. In the case when M is compact and we consider the cohomology with real coefficients[5] $H(M, \mathbb{R})$ the answer can be formulated in terms of the so-called **relative cohomology** $H(\mathfrak{g}, \mathfrak{h}, \mathbb{R})$, which we now define.

By definition, the group of relative k-cochains is $\left(\bigwedge^k(\mathfrak{g}/\mathfrak{h})^*\right)^H$, i.e. consists of H-invariant k-linear antisymmetric maps $c : \mathfrak{g}/\mathfrak{h} \times \cdots \times \mathfrak{g}/\mathfrak{h} \longrightarrow \mathbb{R}$.

The coboundary operator acts as

$$dc(X_0, \ldots, X_k) = \sum_{i<j} (-1)^{i+j} c([X_i, X_j], X_0, \ldots \widehat{X}_i, \ldots \widehat{X}_j \ldots, X_k).$$

Proposition 2. *Let G be a connected compact Lie group, and let H be its closed connected subgroup. Let $\mathfrak{g} = \mathrm{Lie}\,(G)$ and $\mathfrak{h} = \mathrm{Lie}\,(H)$. Then $H^k(G/H, \mathbb{R}) \simeq H^k(\mathfrak{g}, \mathfrak{h}, \mathbb{R})$.* □

This statement can be naturally generalized to the case when $\mathbb{R}$ is replaced by a finite-dimensional vector space V with a linear H-action. Then we can define a G-vector bundle $E = G \times_H V$ and also a relative cohomology group $H^k(\mathfrak{g}, \mathfrak{h}, V)$. It turns out that this group coincides with the k-th Čech cohomology group of G/H with coefficients in the local system of sections of E.

3.4. Cohomology of smooth manifolds.

The general definition of cohomology groups in topology can be formulated as follows. It is a functor from the category $\mathcal{T}op$ of topological spaces to the category $\mathcal{SGR}^\circ$, dual to the category $\mathcal{SGR}$ of supercommutative $\mathbb{Z}$-graded rings.

For smooth compact manifolds the most simple (and most popular) cohomology groups are those with real coefficients. This is a functor from the category $\mathcal{M}an$ of smooth manifolds to the category $\mathcal{SGA}^\circ$, dual to the category $\mathcal{SGA}$ of supercommutative graded algebras over $\mathbb{R}$.

Put simply, to any smooth compact manifold M we associate a graded algebra

$$H^\bullet(M, \mathbb{R}) = \bigoplus_{k=0}^{\dim M} H^k(M, \mathbb{R})$$

and for any smooth map $f : M \longrightarrow N$ we associate the homomorphism

$$f^* : H^\bullet(N, \mathbb{R}) \longrightarrow H^\bullet(M, \mathbb{R})$$

preserving the grading.

[5] See the next section for the definition of real cohomology of a manifold.

There are several equivalent ways to define this cohomology functor. We consider two of them.

The **de Rham cohomology** $H_{DR}(M)$ of a smooth manifold M is defined as follows. The cochain complex is the algebra $\Omega(M)$ of smooth real differential forms on M endowed with the usual grading. The role of the coboundary operator is played by the exterior differential $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$.

The cocycles are the closed forms and the coboundaries are the exact forms.

Example 6. Let $M = S^1$ be the unit circle with the parameter $\theta \in \mathbb{R} \bmod 2\pi\mathbb{Z}$.

The 0-cycles are smooth functions $f(\theta)$ satisfying $f'(\theta) = 0$, i.e. constant functions. Since there are no 0-coboundaries, we see that $H^0_{DR}(S^1) = \mathbb{R}$.

The 1-cycles are all smooth 1-forms $\omega = \phi(\theta)d\theta$ and 1-coboundaries are exact forms $\omega = df = f'(\theta)d\theta$. To decide which cocycles are coboundaries we have to know when a 2π-periodic function $\phi(\theta)$ is a derivative $f'(\theta)$ of some 2π-periodic function f. It is well known that the antiderivative of ϕ is the function $f(\theta) = \int_{\theta_0}^{\theta} \phi(\tau)d\tau$. It is 2π-periodic iff $\int_0^{2\pi} \phi(\theta)d\theta = 0$. So, the coboundary space $B^1(S^1)$ has codimension 1 in the cocycle space $Z^1(S^1)$ and $H^1_{DR}(S^1) = \mathbb{R}$. $\diamondsuit$

Exercise 6. Compute the **Betti numbers** $b_k(M) = \dim H^k_{DR}(M)$

a) for the n-dimensional sphere S^n;

b) for the n-dimensional torus T^n.

Answer: a) $b_k(S^n) = \delta_{k,0} + \delta_{k,n}$;

b) $b_k(T^n) = \binom{n}{k}$.

It follows that S^n can be diffeomorphic to T^n only for $n = 1$. ♣

The definition of the Čech cohomology $H^\bullet(X, \mathbb{R})$ is more involved, but has an important advantage. It makes sense for any topological space and is manifestly invariant under all homeomorphisms.

We start with a covering of our topological space X by a family $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of open subsets. Denote by $U_{\alpha_0 \dots \alpha_k}$ the intersection $\bigcap_{i=0}^k U_{\alpha_i}$.

The k-cochain c associates the real number $c_{\alpha_0 \dots \alpha_k}$ to any $(k+1)$-string of indices $\{\alpha_0 \dots \alpha_k\}$ with non-empty intersection $U_{\alpha_0 \dots \alpha_k}$ so that

$$c_{\alpha_{s(0)} \dots \alpha_{s(k)}} = \operatorname{sign}(s) c_{\alpha_0 \dots \alpha_k} \qquad \text{for any permutation } s \in S_{k+1}.$$

Denote by $C^k(X, \mathcal{U}, \mathbb{R})$ the vector space of all k-cochains and by $C^\bullet(X, \mathcal{U}, \mathbb{R})$ the direct sum of all $C^k(X, \mathcal{U}, \mathbb{R})$.

The coboundary operator is defined by the rule

$$(dc)_{\alpha_0\dots\alpha_{k+1}} = \sum_{i=0}^{k+1}(-1)^i c_{\alpha_0\dots\widehat{\alpha_i}\dots\alpha_{k+1}}$$

where, as usual, the hat means that the corresponding argument must be omitted.

The cohomology of this complex is denoted by $H^\bullet(X, \mathcal{U}, \mathbb{R})$. It may depend in an essential way on the covering $\mathcal{U}$.

Assume that the covering $\mathcal{U}' = \{U'_\beta,\ \beta \in B\}$ is finer than $\mathcal{U} = \{U_\alpha,\ \alpha \in A\}$. It means that every element $U'_\beta \in \mathcal{U}'$ is contained in some element $U_{\alpha(\beta)} \in \mathcal{U}$ where $\beta \mapsto \alpha(\beta)$ is a map from B to A. Then there is a natural map $\phi_{\mathcal{U},\mathcal{U}'}$ from $C^\bullet(X, \mathcal{U}, \mathbb{R})$ to $C^\bullet(X, \mathcal{U}', \mathbb{R})$:

$$\left(\phi_{\mathcal{U},\mathcal{U}'}\, c\right)_{\beta_0\dots\beta_k} = c_{\alpha(\beta_0)\dots\alpha(\beta_k)}.$$

This linear map commutes with the differential, hence gives rise to a map of cohomology which we denote again by $\phi_{\mathcal{U},\mathcal{U}'} : H^\bullet(X, \mathcal{U}', \mathbb{R}) \longrightarrow H^\bullet(X, \mathcal{U}, \mathbb{R})$.

To proceed further, we recall here the categorical definition of a direct or inductive limit. We recommend that the reader compare it with the definition of the direct product in Exercise 4.

Let A be a **directed set**, i.e. a partially ordered set with the property: for any two elements α, β in A there is an element $\gamma \in A$ which is bigger than both α and β.

Suppose that a family $\{X_\alpha\}_{\alpha\in A}$ of objects in $\mathcal{C}$ is given and for every ordered pair $\alpha < \beta$ in A a morphism $\varphi_{\alpha,\beta} \in \mathrm{Mor}_{\mathcal{C}}(X_\alpha,\, X_\beta)$ is defined, so that we have

$$\varphi_{\beta,\gamma} \circ \varphi_{\alpha,\beta} = \varphi_{\alpha,\gamma} \quad \text{for any ordered triple } \alpha < \beta < \gamma \text{ in } A.$$

Consider the auxiliary category $\widetilde{\mathcal{C}}$ defined from the previous data in the following way.

An object of $\widetilde{\mathcal{C}}$ is an object X of $\mathcal{O}b\,\mathcal{C}$ together with a family of morphisms $f_\alpha \in \mathrm{Mor}_{\mathcal{C}}(X_\alpha,\, X)$ for every $\alpha \in A$ such that

$$f_\alpha = f_\beta \circ \varphi_{\alpha,\beta} \quad \text{for any pair } \alpha < \beta.$$

A morphism from $(X,\, \{f_\alpha\})$ to $(Y,\, \{g_\alpha\})$ in the category $\widetilde{\mathcal{C}}$ is a morphism $\xi \in \mathrm{Mor}_{\mathcal{C}}(X,\, Y)$ such that for all $\alpha \in A$ we have $g_\alpha = f_\alpha \circ \xi$.

Assume now that the category $\widetilde{\mathcal{C}}$ has a universal object $(X,\, \{m_\alpha\})$. Then the object X is called the **direct** or **inductive** limit of the family $\{X_\alpha\}$ along

the directed set A and the morphisms m_α are called **canonical maps** of X_α to X.

The dual construction gives the definition of the **inverse** or **projective** limit.

The **Čech cohomology** of a topological space X is defined as a direct limit $H^\bullet_{\check{C}ech}(M, \mathbb{R})$ of graded vector spaces $H^\bullet(X, \mathcal{U}, \mathbb{R})$ along the directed set of all coverings.

This definition is convenient in theory but is highly non-constructive. To get a practical prescription for computing the cohomology, the following result is used.

Leray's Theorem. *Assume that the covering $\mathcal{U}$ has the property: all non-empty intersections $U_{\alpha_0\ldots\alpha_k}$ are contractible sets.*[6] *Then $H^\bullet(X, \mathcal{U}, \mathbb{R}) = H^\bullet_{\check{C}ech}(X, \mathbb{R})$.* □

Example 7. Let us compute the Čech cohomology of the circle $X = S^1$. In this case there is a simple covering of X by three open sets that satisfies the condition of the theorem. Namely, choose any three different points P_0, P_1, P_2 on M and denote by U_k the arc joining P_i with P_j and containing P_k (here $\{i, j, k\}$ is any permutation of $\{0, 1, 2\}$).

The space of 0-cochains consists of triples $(c_0,\ c_1,\ c_2)$ and the space of 1-cochains consists of triples $(c_{0,1},\ c_{1,2},\ c_{2,0})$. The boundary operator is defined by

$$(dc)_{i,j} = c_i - c_j.$$

It is clear that 0-cocycles are the triples of the form $(c,\ c,\ c)$ and 1-coboundaries are the triples $(p,\ q,\ r)$ with $p + q + r = 0$. We get $H^0_{\check{C}ech}(S^1) = \mathbb{R} = H^1_{\check{C}ech}(S^1)$. ◊

Exercise 7. Compute the Čech cohomology of S^n and T^n.

Hint. For S^n use the covering by $n + 2$ open balls such that any $n + 1$ have a non-empty intersection but all $n + 2$ have no common points.

For T^n use the fact that $T^n \simeq S^1 \times \cdots \times S^1$ (n factors). ♣

Exercise 8.*[7] Use the Čech cohomology to prove the following generalization of the famous Helly theorem on convex sets.

Claim. *Assume that a family $\{U_k\}_{0\le k\le n+1}$ of $n+2$ open subsets in $\mathbb{R}^n$ has the property: all m-intersections $U_{k_1\ldots k_m} := \bigcap_{i=1}^m U_{\alpha_{k_i}}$ are non-empty and contractible for $1 \le m \le n+1$. Then all $n + 2$ sets $\{U_k\}_{0\le k\le n+1}$ have a common point.* ♣

[6] Cf. the definition of Leray atlas in Appendix II.1.2.

[7] The idea of this exercise is due to D. Kazhdan.

We see in the examples above that de Rham and Čech cohomology of spheres and tori are the same. It is not a coincidence.

de Rham's Theorem. *For any compact smooth manifold there is a canonical isomorphism*

$$H^k_{DR}(M) \simeq H^k_{\check{C}ech}(M, \mathbb{R}).$$

Sketch of the proof. Let $c \in H^k_{DR}$, and let $\omega \in \Omega^k(M)$ represent the class c. Consider a Leray covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and for any $m \geq 0$ construct the family of closed $(k-m)$-forms $\omega_{\alpha_0,\ldots,\alpha_m}$ on $U_{\alpha_1,\ldots,\alpha_m}$ as follows.

1. Denote by ω_α the restriction of ω to U_α.

2. Since ω is closed and U_α is contractible, we have $\omega_\alpha = d\theta_\alpha$ for some $(k-1)$-form θ_α on U_α. Put $\omega_{\alpha,\beta} = \theta_\alpha - \theta_\beta$ on $U_{\alpha,\beta}$.

3. The form $\omega_{\alpha,\beta}$ is a closed $(k-1)$-form because $d\omega_{\alpha,\beta} = d(\theta_\alpha - \theta_\beta) = \omega_\alpha - \omega_\beta = 0$. Since $U_{\alpha,\beta}$ is contractible, the form $\omega_{\alpha,\beta}$ is also exact and we can write it as $d\theta_{\alpha,\beta}$ for some $(k-2)$-form $\theta_{\alpha,\beta}$ on $U_{\alpha,\beta}$. Then we define the $(k-2)$-form $\omega_{\alpha,\beta,\gamma}$ on $U_{\alpha,\beta,\gamma}$ as $\theta_{\alpha,\beta} + \theta_{\beta,\gamma} + \theta_{\gamma,\alpha}$. It is closed because $d\omega_{\alpha,\beta,\gamma} = d\theta_{\alpha,\beta} + d\theta_{\beta,\gamma} + d\theta_{\gamma,\alpha} = \omega_{\alpha,\beta} + \omega_{\beta,\gamma} + \omega_{\gamma,\alpha} = 0$, etc.

Eventually we come to the family of closed 0-forms $\omega_{\alpha_0,\ldots,\alpha_k}$ on $U_{\alpha_0,\ldots,\alpha_k}$. But a closed 0-form on a connected domain is just a constant function, i.e. a real number. Moreover, by construction, the numbers $\omega_{\alpha_0,\ldots,\alpha_k}$ are antisymmetric with respect to permutation of indices. So, we get a Čech cochain $c_{\alpha_0,\ldots,\alpha_k} = \omega_{\alpha_0,\ldots,\alpha_k}$ in $C^k_{\check{C}ech}(M, \mathcal{U}, \mathbb{R})$.

We omit the verification that this k-cochain c is actually a cocycle and its cohomology class $[c]$ depends only on the class $[\omega]$. So, we have defined a map $H^k_{DR}(M, \mathbb{R}) \longrightarrow H^k_{\check{C}ech}(M, \mathbb{R})$.

For the definition of the inverse map $H^k_{\check{C}ech}(M, \mathbb{R}) \simeq C^k_{\check{C}ech}(M, \mathcal{U}, \mathbb{R}) \longrightarrow H^k_{DR}(M, \mathbb{R})$ we need a partition of unity $\{\phi_\alpha\}$ subordinated to the covering $\mathcal{U}$. Using this partition and starting from a Čech k-cocycle $c_{\alpha_0,\ldots,\alpha_k}$, we construct for any $m \geq 0$ the family of closed m-forms $\omega_{\alpha_0,\ldots,\alpha_{k-m}}$ on $U_{\alpha_0,\ldots,\alpha_{k-m}}$.

As $\omega_{\alpha_0,\ldots,\alpha_k}$ we take just the family of constants $c_{\alpha_0,\ldots,\alpha_k}$.

Then we define the m-form $\omega_{\alpha_0,\ldots,\alpha_{k-m}}$ on $U_{\alpha_0,\ldots,\alpha_{k-m}}$ by the formula

$$\omega_{\alpha_0,\ldots,\alpha_{k-m}} = \sum_{\alpha_{k-m+1},\ldots,\alpha_k} c_{\alpha_0,\ldots,\alpha_k} d\phi_{\alpha_{k-m+1}} \wedge \cdots \wedge d\phi_{\alpha_k}.$$

Finally, we associate to a cocycle c the k-form ω such that

$$\omega\,\Big|_{U_{\alpha_0}} = \sum c_{\alpha_0,\ldots,\alpha_k} d\phi_{\alpha_1} \wedge \cdots \wedge d\phi_{\alpha_k}.$$

One can check that ω is closed and the correspondence $[c] \mapsto [\omega]$ is the inverse to the map $[\omega] \mapsto [c]$ constructed above. $\square$

In conclusion we mention the connection between the homology and homotopy groups.

Proposition 3. *If $\pi_k(X) = 0$ for $k < n$, then*

$$H_n(X, \mathbb{Z}) = \begin{cases} \pi_n(X) & \textit{for } n > 1 \textit{ or } n = 0, \\ \pi_1(X)/[\pi_1(X), \pi_1(X)] & \textit{for } n = 1. \end{cases}$$

Appendix II

Smooth Manifolds

1. Around the definition

In this appendix we collect some information about smooth manifolds in the form used in the book.

There are three different ways of thinking about manifolds: geometric, analytic, and algebraic. It is important to combine all three and to be able to switch from one to another. As an illustration, I will tell you an old joke about two math students who met after one had a geometry class and the other had an algebra class.

First student: I have finally understood why the system of linear equations has a unique solution: it is because two lines intersect in one point!

Second student: And I have finally understood why two lines intersect in one point: it is because a system of linear equations has a unique solution!

The conclusion for a reader: there are three (not just two!) important facts:

1. Two straight lines intersect in one point.

2. A system of two linear equations with two unknowns has a unique solution.

3. The two statements above are actually the same.

1.1. Smooth manifolds. Geometric approach.

Smooth manifolds appear in geometry as collections of geometric objects: points, vectors, curves, etc. (or equivalence classes of these objects).

Usually, these collections come together with a natural topology: we can define what a neighborhood of a given geometric object m is in a given collection M.

The characteristic property of a manifold is the ability to describe the position of a point $m \in M$ at least locally by an n-tuple of real numbers $(x^1, x^2, \ldots, x^n)$. These numbers are called **local coordinates** and identify a neighborhood of $m \in M$ with some open domain in $\mathbb{R}^n$.

The first important class of manifolds is formed by **smooth submanifolds** in Euclidean spaces. A smooth submanifold $M \subset \mathbb{R}^N$ can be given as the set of all solutions to a system of equations

$$F_k(x) = 0, \quad 1 \le k \le s, \tag{1}$$

where F_k are smooth real functions on $\mathbb{R}^N$.

Geometrically, the collection of function $\{F_k,\ 1 \le k \le s\}$ can be viewed as a smooth map $F : \mathbb{R}^N \longrightarrow \mathbb{R}^s$ and the set (1) is precisely the preimage $F^{-1}(0)$ of a point $\{0\} \in \mathbb{R}^s$.

In fact, not every system of type (1) defines a manifold. A simple sufficient condition is provided by

Proposition 1. *Assume that the following condition is satisfied:*

(2) *The rank of the $s \times N$ matrix $DF = \left\| \dfrac{\partial F_k}{\partial x^j} \right\|$ is equal to a constant r in a neighborhood of M.*

Then the set M defined by the system of equations (1) *is a smooth manifold of dimension $n = N - r$.*

A local coordinate system in a neighborhood of a given point is provided by a projection of M on a suitable n-dimensional coordinate subspace in $\mathbb{R}^N$.

Scheme of the proof. For a given point $x \in M$ let us choose the indices $i_1, \ldots, i_r$ so that the submatrix formed by corresponding columns of $DF(x)$ is of rank r. Then the Implicit Function Theorem ensures that in some neighborhood $U(x) \subset M$ the coordinates $x_{i_1}, \ldots, x_{i_r}$ can be expressed as smooth functions of the remaining n coordinates. $\square$

Remark 1. Actually, this type of manifold is a general one: according to the famous Whitney theorem, every abstract[1] smooth n-dimensional manifold M can be realized as a submanifold in $\mathbb{R}^{2n}$.

[1] See the next section for the definition of an abstract manifold.

Moreover, if M is a compact n-dimensional manifold, then randomly chosen $2n+1$ smooth functions on M give an embedding of M in $\mathbb{R}^{2n+1}$. (The precise meaning of this informal statement is that for $k \geq 2$ the set of embeddings is open and dense in the space of all k-smooth maps $\phi : M \longrightarrow \mathbb{R}^{2n+1}$ endowed with a suitable topology.) ♡

Example 1. Let M_c be the subset in $\mathrm{Mat}_n(\mathbb{R})$ defined by the equation

$$\det A = c.$$

We show that for all $c \neq 0$ the set M_c is a smooth submanifold in $\mathrm{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. For this, according to Proposition 1, we have to check that the partial derivatives

$$\widehat{A}_{ij} := \frac{\partial}{\partial A_{ji}} \det A$$

form a non-zero matrix $\widehat{A}$. But from linear algebra we know that $A \cdot \widehat{A} = \det A \cdot 1$. Therefore, for $c \neq 0$ and $A \in M_c$ we have $\widehat{A} = c \cdot A^{-1} \neq 0$. ♢

There is another, more general, way to construct a manifold geometrically. Let X be some set of geometric objects and consider a quotient set $M = X/\sim$ with respect to some equivalence relation $\sim$.

Often this equivalence relation is defined via the action of some group G on X: two points of X are equivalent if they belong to the same G-orbit. As coordinates on the set of equivalence classes we can use G-invariant functions on X.

Proposition 2. *Let X be a smooth N-dimensional manifold, and let G be a group of smooth transformations of X. Assume that all G-orbits in X are closed smooth submanifolds of the same dimension k. Then the set of orbits $M = X/G$ is a smooth $(N-k)$-dimensional manifold, called the* **quotient** *or* **factor** *manifold.*

Scheme of the proof. Let $x \in X$. Consider an $(N-k)$-dimensional submanifold $S \subset X$ passing through x and transversal to the G-orbit Ω_x. Then for a sufficiently small neighborhood $U \ni x$ all G-orbits Ω passing through U intersect $S \cap U$ exactly in one point (the Implicit Function Theorem applied to the map $G \times S \longrightarrow X$). So, the set of such orbits is identified with $S \cap U$ and provides a local coordinate system on $M = X/G$. □

Warning. It seems that this proof does not use the assumption that the orbits are closed submanifolds. Actually, this assumption is needed to show that M/G is Hausdorff (see the next section).

Recall the standard example when the assumption does not hold: X is the torus $\mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the action of the group $G = \mathbb{R}$ by the formula

$$t \cdot (x, y) = (x + t, y + \alpha t)$$

where $x \bmod 1$, $y \bmod 1$ are the usual coordinates on $\mathbf{T}^2$. If α is rational, the orbits are circles and X/G is a smooth manifold (actually, also a circle). But when α is an irrational real number, all G-orbits are everywhere dense in X and X/G is not Hausdorff. ♠

Example 2. The n-dimensional real projective space $\mathbb{P}^n(\mathbb{R})$ is the quotient of the set $X = \mathbb{R}^{n+1}\backslash\{0\}$ modulo the equivalence relation:

$$x \sim y \Longleftrightarrow x \text{ is proportional to } y \text{ (i.e. } x = c \cdot y \text{ for some real number } c).$$

Clearly the equivalence classes are just the orbits of the multiplicative group $\mathbb{R}^\times$ acting on X by dilation-reflections. Note also that $\mathbb{P}^n(\mathbb{R})$ can be viewed geometrically as the set of all 1-dimensional subspaces in $\mathbb{R}^{n+1}$.

Denote by $[x]$ the equivalence class of $x \in X$ and by $(x^0, x^1, \ldots, x^n)$ the coordinates of x. Then a point $[v] \in \mathbb{P}^n$ is specified by the so-called **homogeneous coordinates** $(x^0 : x^1 : \cdots : x^n)$ where the colons are used to show that only the ratios of the coordinates matter.

Let U_i be the part of $\mathbb{P}^n$ where $x^i \neq 0$. On U_i we introduce n local coordinates

$$u^j_{(i)} = \frac{x^j}{x^i}, \qquad 0 \leq j \leq n,\ j \neq i.$$

The transition functions between the two coordinate systems on $U_i \cap U_k$ are rational with non-vanishing denominators:

$$u^j_{(k)} = u^j_{(i)}/u^k_{(i)} \text{ for } j \neq i,\, k, \qquad u^i_{(k)} = 1/u^k_{(i)}.$$

One can embed $\mathbb{P}^n(\mathbb{R})$ into $\mathbb{R}^N$, $N = \frac{(n+1)(n+2)}{2}$, using the map

$$x \mapsto \left\{ \frac{x^i\, x^j}{\sum_{k=0}^n (x^k)^2} \right\}_{0 \leq i \leq j \leq n}.$$

♢

Combining the notions of submanifold and of quotient manifold we can introduce a manifold structure on many sets of geometric origin. This is the reason why we call this approach the "geometric definition of a manifold".

1.2. Abstract smooth manifolds. Analytic approach.

In calculus courses we usually consider smooth functions defined on Euclidean spaces $\mathbb{R}^n$ or on open domains $D \subset \mathbb{R}^n$. The crucial role is played by coordinate systems that allow us to consider a function f on D just as a function of n real variables. There exist more complicated sets that also admit coordinate systems at least locally. They are called **abstract manifolds** and we give their definition below.

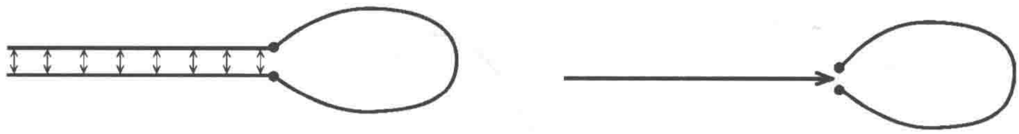

Figure 1

Definition 1. A **k-smooth n-dimensional manifold** is a topological space M that admits a covering by open sets U_α, $\alpha \in A$, endowed with one-to-one continuous maps $\phi_\alpha \colon U_\alpha \to V_\alpha \subset \mathbb{R}^n$ so that all maps $\varphi_{\alpha,\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ are k-smooth (i.e. have continuous partial derivatives of order $\leq k$) whenever defined.

We call the manifold M **smooth** (resp. **analytic**) when all the maps $\varphi_{\alpha,\beta}$ are infinitely differentiable (resp. analytic).

The following terminology is usually used:

- the sets U_α are called **charts**;
- the functions $x^i_\alpha = x^i \circ \phi_\alpha$ are called **local coordinates**;[2]
- the functions $\varphi_{\alpha,\beta}$ are called the **transition functions**;
- the collection $\{U_\alpha,\, \phi_\alpha\}_{\alpha\in A}$ is called the **atlas** on M.

Remark 2. For technical reasons two additional conditions are usually imposed.

1. A topological space is called **Hausdorff** if any two different points have non-intersecting neighborhoods.

2. A manifold is called **separable** if it can be covered by a countable system of charts.

In our book we always assume that both conditions are satisfied unless the opposite is explicitly formulated.

An example of a non-Hausdorff (and non-orientable!) 1-dimensional manifold can be seen in Figure 1.

Note also that the sets of orbits for a non-compact Lie group are usually non-Hausdorff. ♡

Definition 1 above still needs some comments. Namely, a given topological space M can be endowed with several different atlases satisfying all the requirements of the definition. Should we consider the corresponding objects as different or as the same manifold? To clear up the situation, we introduce

Definition 2. Two atlases $\{U_\alpha\}_{\alpha\in A}$ and $\{U'_\beta\}_{\beta\in B}$ are called **equivalent** if the transition functions from any chart of the first atlas to any chart of the

[2]Here $\{x^i\}_{1\leq i\leq n}$ are the standard coordinates in $\mathbb{R}^n$;

second one are k-smooth (whenever defined). The **structure of a smooth manifold** on M is an equivalence class of atlases.

Remark 3. When dealing with a smooth manifold, we usually follow a common practice and each time use only one, the most appropriate, atlas keeping in mind that we can always replace it by an equivalent one.

In practical computations people usually prefer minimal atlases. For example, for the n-dimensional sphere S^n and for the n-dimensional torus T^n there exist atlases with only two charts.

Sometimes it is convenient to assume that each chart is **contractible** (i.e. homeomorphic to an open ball). An interesting geometric characteristic of a manifold is the cardinality of a minimal atlas with contractible charts. (For example, for S^n it is equal to 2 while for T^n it is $n+1$.)

In homology theory the special role is played by the so-called **Leray atlases** that have the property: all non-empty finite intersections $\bigcap_{k=1}^{n} U_k$ are contractible. The existence of such atlases is not evident but it is known that any compact smooth manifold admits a finite Leray atlas. For example, one can use convex open sets with respect to some Riemannian metric.

The maximal atlases are also useful in some theoretical questions. The reason is that in each equivalence class there is exactly one maximal atlas. It consists of all charts that have smooth transition functions with all charts of any atlas from the given class. ♡

If M_1, M_2 are k-smooth manifolds, then for any $m \leq k$ there is a natural notion of an **m-smooth** map $f : M_1 \to M_2$. Namely, for any point $x_0 \in M_1$ and any charts $U \ni x$, $V \ni f(x_0)$ the local coordinates of $f(x)$ in V must be m-smooth functions of local coordinates of x in U.

A smooth map from one manifold to another which has a smooth inverse map is called a **diffeomorphism**. Two manifolds are called **diffeomorphic** if there is a diffeomorphism from one to another.

Warning. Different smooth manifold structures on a given topological space M can define diffeomorphic manifolds. For instance, any continuous invertible map $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ defines a one-chart atlas, hence a smooth manifold structure on $\mathbb{R}$.

All smooth manifolds obtained in this way are diffeomorphic to the standard $\mathbb{R}$. On the other hand, the atlases corresponding to functions ϕ and ψ are equivalent only if $\phi \circ \psi^{-1}$ is smooth. ♠

Remark 4. For topological (i.e. 0-smooth) manifolds M of small dimensions the following phenomenon is observed:

there is exactly one (up to diffeomorphism) smooth manifold $\widetilde{M}$ that is homeomorphic to M.

This statement fails in higher dimensions. The first counterexample for compact manifolds is the 7-dimensional sphere S^7, which is homeomorphic to 28 different smooth manifolds.

The first non-compact counterexample is still more intriguing. There exists a smooth manifold that is homeomorphic but not diffeomorphic to the standard space $\mathbb{R}^4$.

Finally, there exist topological manifolds that are homeomorphic to no smooth manifold at all. The simplest known compact example has dimension 8. ♡

We conclude this section by introducing the notion of orientation. Two charts with coordinates $\{x^i\}$ and $\{y^j\}$ are called **positively related** if the following condition is satisfied:

$$\det \left\| \frac{\partial y^j}{\partial x^i} \right\| > 0 \quad \text{whenever defined.} \tag{3}$$

We say that two charts are **negatively related** if the opposite inequality holds.

Warning. It can happen that two charts are both positively and negatively related (when they are disjoint), or neither positively nor negatively related (when their intersection is not connected; see Example 3 below, the case $n = 2$). ♠

Definition 3. A manifold M is called **orientable** if it admits an atlas with positively related charts. We call such an atlas **oriented**, as well as a manifold endowed with it. The choice of an oriented atlas is called an **orientation** of M.

Example 3 (Möbius band). Let us consider n copies[3] of the open rectangle $0 < x < 3$, $0 < y < 1$ and glue them together as follows.

Let x_k, y_k denote the coordinates on the k-th rectangle. We identify the part $2 < x_k < 3$ of the k-th rectangle with the part $0 < x_{k+1} < 1$ of the $(k+1)$-st rectangle according to the transition functions

$$x_{k+1} = x_k - 2, \qquad y_{k+1} = y_k, \qquad 1 \le k \le n-1,$$

and the part $2 < x_n < 3$ of the n-th rectangle with the part $0 < x_1 < 1$ of the first rectangle according to the transition functions

$$x_1 = x_n - 2, \qquad y_1 = -y_n.$$

[3] The description below makes sense for $n \ge 2$. In the case $n = 1$ it should be slightly modified. We leave the details to the reader.

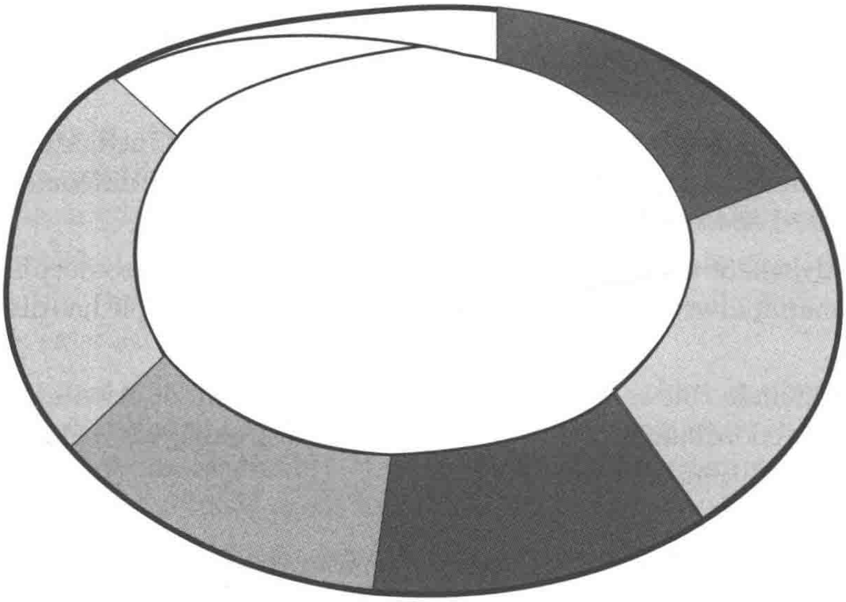

Figure 2. Möbius band.

The resulting 2-dimensional manifold (which can be easily manufactured from n pieces of paper) is the simplest example of a **non-orientable** manifold. $\diamondsuit$

Exercise 1. Verify that the manifold $\mathbb{P}^n(\mathbb{R})$ is orientable iff $n = 0$ or n is odd.

Hint. Compute the Jacobian, i.e. the determinant of the Jacobi matrix

$$J = \left\| \frac{\partial u^i_{(k)}}{\partial u^j_{(l)}} \right\|^{i \neq k}_{j \neq l}$$

for the standard charts $U^{(k)}$, $U^{(l)}$ (see Example 2).

Answer: $J = -\left(u^l_{(k)}\right)^{n+1}$. ♣

Proposition 3. *Any connected orientable manifold of positive dimension admits exactly two, up to equivalence, oriented atlases (hence, two different orientations).*

Proof. Indeed, for any oriented atlas $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ there exists an atlas $\mathcal{U}' = \{U'_\alpha\}_{\alpha \in A}$ whose charts are negatively related with charts of the first atlas (just change the sign of one local coordinate in each chart). Now let $\mathcal{V} = \{V_\gamma\}_{\gamma \in \Gamma}$ be any oriented atlas. Any point $m \in M$ is covered by some chart $U_{\alpha(m)} \in \mathcal{U}$ and by some chart $V_{\gamma(m)} \in \mathcal{V}$. Define the function f by the

rule:

$$f(m) = \begin{cases} 1 & \text{if } U_{\alpha(m)} \text{ and } V_{\gamma(m)} \text{ are positively related,} \\ -1 & \text{if } U_{\alpha(m)} \text{ and } V_{\gamma(m)} \text{ are negatively related.} \end{cases}$$

We leave it to the reader to check that this function is correctly defined (i.e. does not depend on the choice of the charts covering m) and is locally constant. Since M is connected, f must be constant.

It follows that the atlas $\mathcal{V}$ is equivalent to one of the atlases $\mathcal{U}$, $\mathcal{U}'$. □

1.3. Complex manifolds.

The analytic definition of a smooth real manifold has a natural modification. Namely, let us consider the charts with complex coordinates $z^1, \dots, z^n$ and require that the transition functions $\phi_{\alpha,\beta}$ be 1-smooth in the complex sense (which implies that they are analytic). What we obtain is the definition of a **complex manifold**.

Example 4. The complex projective space $\mathbb{P}^n(\mathbb{C})$ is defined exactly in the same way as $\mathbb{P}^n(\mathbb{R})$ but with complex homogeneous coordinates $(z^0 : z^1 : \dots : z^n)$. We use the fact that the transition functions are rational, hence analytic in their domains of definition. ♢

Of course, any complex n-dimensional manifold can be viewed as a real manifold of dimension $2n$. The converse is far from being true.

First, not all even-dimensional real manifolds admit a complex structure. For example, on a sphere S^{2n} such a structure exists only for $n = 1$.

On the other hand, there could be many non-equivalent complex manifolds that are diffeomorphic as real manifolds.

Example 5. Let T^2 be the 2-torus. Every complex structure on it comes from the realization of T^2 as the quotient manifold $\mathbb{C}/L_\tau$ where $L_\tau = \mathbb{Z} + \tau \cdot \mathbb{Z}$ is a lattice (discrete subgroup) in $\mathbb{C}$ generated by 1 and $\tau \in \mathbb{C} \backslash \mathbb{R}$.

It is known that all complex analytic automorphisms of $\mathbb{C}$ are affine transformations. It follows that $\mathbb{C}/L_{\tau_1}$ and $\mathbb{C}/L_{\tau_2}$ are equivalent as complex manifolds iff

$$\tau_2 = \frac{m\tau_1 + n\tau_1}{p\tau_1 + q\tau_1} \qquad \text{where } \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in GL(2, \mathbb{Z}).$$

So, the set of all complex manifolds homeomorphic to T^2 is itself a complex manifold, the so-called **moduli space** $M_1 = (\mathbb{C} \backslash \mathbb{R})/GL(2, \mathbb{Z})$. ♢

Warning. Note that the analogue of the Whitney theorem no longer holds for complex manifolds. For example, no complex compact manifolds of positive dimension can be smoothly embedded in $\mathbb{C}^N$. In the simplest case

$M = \mathbb{P}^1(\mathbb{C})$ it follows from the well-known fact: any bounded holomorphic function on $\mathbb{C}$ is constant.

One can try to realize a general complex manifold as a submanifold of $\mathbb{P}^N(\mathbb{C})$ for large N. But this endeavor also fails: according to the Kodaira theorem all compact complex submanifolds of $\mathbb{P}^N(\mathbb{C})$ are **algebraic**, i.e. given by a system of polynomial equations. Therefore, even complex tori $T^n = \mathbb{C}^n/L$, where L is a lattice of rank $2n$ in $\mathbb{C}^n$, can be embedded in a projective space only for very special lattices L. ♠

1.4. Algebraic approach.

There is a more algebraic way to define a smooth manifold M. Namely, let $C^\infty(M)$ be the algebra of smooth functions on M. The **support** of a function f, denoted by $\operatorname{supp} f$, is by definition the closure of the set $\{m \in M \mid f(m) \neq 0\}$.

Denote by $\mathcal{A}(M)$ (the notation $C_c^\infty(M)$ is also used) the algebra of all smooth real functions with compact support on M. In particular, if M itself is compact, then $\mathcal{A}(M)$ is the algebra of all smooth functions on M.

A remarkable fact is that a smooth manifold M is completely determined by the algebraic structure of $\mathcal{A}(M)$.

Theorem 1. *Any non-zero $\mathbb{R}$-algebra homomorphism $\chi : \mathcal{A}(M) \to \mathbb{R}$ has the form*

$$\chi = \chi_m : f \mapsto f(m) \quad \textit{for some } m \in M. \tag{4}$$

□

So, we can reconstruct the set M from the algebra $\mathcal{A}$ as a set of all non-zero algebra homomorphisms of $\mathcal{A}$ to $\mathbb{R}$.

It turns out that smooth maps from one smooth manifold to another can also be described algebraically. In the case of compact manifolds the situation is very simple: smooth maps $\phi : M \longrightarrow N$ correspond to non-zero algebra homomorphisms $\Phi : \mathcal{A}(N) \to \mathcal{A}(M)$ by the formula (5) below. To include the case of non-compact manifolds we need two additional definitions.

1. A smooth map $\phi : N \longrightarrow M$ is called **proper** if the preimage of any compact set $K \subset M$ is a compact subset in N.

This condition is automatically satisfied if N is compact. Note that there are no proper maps from a non-compact manifold to a compact one.

2. An algebra homomorphism $\Phi : A \longrightarrow B$ is called **essential** if its image $\Phi(A)$ is not contained in any ideal of B different from B itself.

If A has a unit, then Φ is essential iff B also has a unit and Φ is a unital homomorphism.

In particular, there is no essential homomorphism from an algebra with unit to an algebra without unit.

Theorem 2. *Let M and N be any smooth manifolds. Then the relation*

$$\Phi(f) = f \circ \phi \tag{5}$$

establishes a bijection between the proper maps $\phi : N \longrightarrow M$ and the essential algebra homomorphisms $\Phi : \mathcal{A}(M) \to \mathcal{A}(N)$.

In particular, there is a bijection between points of M and non-zero homomorphisms $\phi : \mathcal{A}(M) \to \mathbb{R}$. □

The main step in the proofs of Theorems 1 and 2 is

Lemma 1. *Let $\chi : \mathcal{A}(M) \to \mathbb{R}$ be a non-zero homomorphism. Consider the ideal $I_\chi \subset \mathcal{A}(M)$, which is the kernel of χ. Then there exists a point $m \in M$ such that all functions $f \in I_\chi$ vanish at m.*

Proof. Assume that there is no such point. Then for any $m \in M$ there is a function $f_m \in I_\chi$ such that $f_m(m) \neq 0$. But in this case f_m is also non-vanishing in some neighborhood U_m of m. Let $g \in \mathcal{A}(M)$ be an arbitrary function. Since the set $S = \operatorname{supp} g$ is compact, we can choose finitely many points $m_1, \ldots, m_N$ so that the corresponding neighborhoods U_{m_i}, $1 \leq i \leq N$, cover all of S. Then the function $F(x) := \sum_i f_{m_i}^2(x)$ belongs to I_χ and is everywhere positive on S. Hence, $\frac{g}{F} \in \mathcal{A}(M)$ and $g = F \cdot \frac{g}{F} \in I\chi$, a contradiction to $\chi \neq 0$. □

In conclusion we give an example of a non-essential homomorphism.

Example 6. Let $A = \mathcal{A}(\mathbb{R})$, $B = \mathcal{A}(S^1)$. Choose on $\mathbb{R}$ the real coordinate x and on S^1 the complex coordinate z, $|z| = 1$. For $f \in \mathcal{A}(\mathbb{R})$ we put

$$\Phi(f)(z) = f\left(i \cdot \frac{1-z}{1+z}\right) = f\left(\tan \frac{t}{2}\right) \quad \text{if } z = e^{it},\ |t| < \pi.$$

Then the image $\Phi(\mathcal{A}(\mathbb{R}))$ consists of functions on S^1 that vanish in some neighborhood of the point $P = -1$ and is a non-trivial ideal in $\mathcal{A}(S^1)$. So, Φ is non-essential.

The homomorphism Φ does not correspond to any map $\phi : S^1 \longrightarrow \mathbb{R}$. Indeed, the map $\chi_P \circ \Phi$ is identically zero, hence the image $\phi(P)$ is not defined. ◊

The algebraic definition given above can be formulated in categorical language as follows. Let $\mathcal{A}lg\,(K)$ be the category whose objects are commutative associative algebras over a field K and whose morphisms are K-algebra homomorphisms. Then the category $\mathcal{M}an$ of smooth manifolds is just a subcategory of $\mathcal{A}lg\,(\mathbb{R})^{\circ}$.

It is tempting to consider all objects of $\mathcal{A}lg\,(K)^{\circ}$ as generalized manifolds. In other words, the following principle is proclaimed:

Any commutative associative algebra $\mathcal{A}$
is an algebra of functions on some "manifold" $\mathrm{Spec}(\mathcal{A})$.

This principle suggests the geometric interpretations of many other pure algebraic notions, e.g.,

projective $\mathcal{A}$-module $\mathcal{L}$ $\longleftrightarrow$ space of sections of a vector bundle L over $\mathrm{Spec}(\mathcal{A})$;

Lie algebra *Vect* $(\mathrm{Spec}(\mathcal{A}))$ $\longleftrightarrow$ Lie algebra of derivations of $\mathcal{A}$; etc.

This point of view has been basic to modern algebraic geometry since the seminal works by Grothendieck.

Last time it became popular to also consider non-commutative algebras in this context (non-commutative algebraic geometry, or the theory of non-commutative manifolds). Two special cases are of greatest importance because of the role they play in modern mathematical physics:

a) supermanifolds, where the set of functions forms a $\mathbb{Z}_2$-graded supercommutative algebra;

b) quantum groups, which can be defined as group-like objects in the category of non-commutative manifolds.

We only briefly mention them in our lectures but the extension of the orbit method to these new domains is a very challenging problem that is only partly resolved (see, e.g., [**QFS**], [**So**], [**Ka2**]).

2. Geometry of manifolds

We assume that the reader is acquainted with the elements of differential geometry. Here we only recall some notions and facts using all three approaches: geometric, analytic, and algebraic, which were described above.

2.1. Fiber bundles.

A manifold X is called a **fiber bundle** over a **base** B with a **fiber** F if a smooth surjective map $p: X \longrightarrow B$ is given such that locally X is a direct product of a part of the base and the fiber.

More precisely, we require that any point $b \in B$ has a neighborhood U_b such that $p^{-1}(U_b)$ can be identified with $U_b \times F$ via a smooth map α_b so that

the following diagram (where p_1 denotes the projection to the first factor) is commutative:

$$\begin{array}{ccc} p^{-1}(U_b) & \xrightarrow{\alpha_b} & U_b \times F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_1} \\ U_b & = & U_b. \end{array}$$

From this definition it follows that all sets $F_b := p^{-1}(b)$, $b \in B$, called **fibers**, are smooth submanifolds diffeomorphic to F.

We use the notation

$$(X, B, F, p) \qquad \text{or} \qquad F \longrightarrow X \xrightarrow{p} B \qquad \text{or else} \qquad X \xrightarrow{F} B$$

for a fiber bundle with the total space X, base B, fiber F, and projection p.

Sometimes fiber bundles are called **twisted** or **skew** products of B and F. The reason is that a direct product $X = B \times F$ is a particular kind of fiber bundle: one can put $U = B$, $\alpha = \mathrm{Id}$. We observe also that B and F play non-symmetric roles in the construction of a skew product.

The collection of all fiber bundles forms a category where the morphisms from $X_1 \xrightarrow{F_1} B_1$ to $X_2 \xrightarrow{F_2} B_2$ are pairs of smooth maps $(f : X_1 \longrightarrow X_2,\ \phi : B_1 \longrightarrow B_2)$ such that the following diagram is commutative:

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{\phi} & B_2. \end{array}$$

The objects that are equivalent to a direct product are called **trivial bundles**.

A map $s : B \longrightarrow X$ is called a **section** of a fiber bundle $X \xrightarrow{F} B$ if $p \circ s = \mathrm{Id}$. In the case of a trivial bundle the sections can be identified with functions $f : B \longrightarrow F$. So, the sections of a fiber bundle $X \xrightarrow{F} B$ give a natural generalization of the notion of a function from B to F.

Example 7. The subcategory of fiber bundles with given base S^1 and given fiber $\mathbb{R}^1$ has only two equivalence classes of objects: the cylinder $S^1 \times \mathbb{R}^1$ (a trivial bundle) and the Möbius band M (a non-trivial bundle). $\diamondsuit$

A fiber bundle (X, B, F, p) is called a **vector bundle** if F and all F_b, $b \in B$, are vector spaces (real or complex) and all maps α_b, $b \in B$, are linear on the fibers.

We now consider in more detail the tangent bundle TM, the most important example, which was the source of the whole theory of vector bundles.

A **tangent vector** ξ to a manifold M at the point $m \in M$ can be defined in three different ways.[4]

a) Geometrically, as an equivalence class of smooth parametrized curves in M passing through m. Namely, two curves $x(t)$ and $y(t)$ are called equivalent if

1. $x(0) = y(0) = m$;
2. $|x(t) - y(t)| = o(t)$ in any local chart covering m.

b) Analytically, as an expression $\sum_{i=1}^n a^i \frac{\partial}{\partial x^i}$ in some local coordinate system centered at m.

c) Algebraically, as a linear functional ξ on $\mathcal{A}(M)$ satisfying the condition

$$\xi(fg) = \xi(f)g(m) + f(m)\xi(g). \tag{6}$$

The explicit correspondence between these definitions looks as follows. A vector $a = \{a^i\} \in \mathbb{R}^n$ corresponds:

a) to the equivalence class of curves $x(t)$ for which $\frac{dx^i}{dt}(0) = a^i$;

b) to the expression $\sum_{i=1}^n a^i \frac{\partial}{\partial x^i}$;

c) to the linear functional $f \mapsto \sum_{i=1}^n a^i \frac{\partial f}{\partial x^i}(m)$.

We denote by $T_m M$ the set of all tangent vectors to M at m and call it the **tangent space**. The union $TM := \bigcup_{m\in M} T_m M$ forms a fiber bundle over M with the projection p that sends all elements of $T_m M$ to m.

Indeed, if $U \subset M$ is a local chart with coordinates $(x^1, \ldots, x^n)$, then any tangent vector $a \in T_x M$ can be written as $a = a^k \partial_k$ where $a^k = a^k(x)$ are numerical coefficients and $\partial_k := \partial/\partial x^k$ denotes the partial derivative (which is a tangent vector in the sense of the third definition).

So, the set $p^{-1}(U)$ is identified with $U \times \mathbb{R}^n$. It is by definition a local chart on TM with coordinates $(x^1, \ldots, x^n; a^1, \ldots, a^n)$.

The (smooth) sections of TM are called (smooth) **vector fields** on M. They also admit several interpretations:

a) Geometric: as generators of flows, i.e. one-parametric groups $\{\Phi_t\}$ of diffeomorphisms of M defined by the ordinary differential equation (where the dot denotes the time derivative)

$$\dot{\Phi}_t(m) = \xi(m).$$

Here one has to suppose M is compact or restrict the behavior of the vector field at infinity: otherwise the flow $\{\Phi_t\}$ exists only locally.

[4]We suggest that the reader compare the mathematical definitions below with the physical definition of a tangent vector as a **velocity** of a point moving along a given surface.

b) Analytic: as differential operators $\sum a^i(x)\frac{\partial}{\partial x^i}$ in each local coordinate system on M with the natural transformation rule under the coordinate changing.

Namely, the vector field $\sum a^i(x)\frac{\partial}{\partial x^i}$ in another chart with local coordinates $\{y^j,\ 1 \leq j \leq n\}$ looks like $\sum_j \tilde{a}_j(y)\frac{\partial}{\partial y^j}$ where $\tilde{a}^j(y) = \sum_i \frac{\partial y^j}{\partial x^i} a^i(x(y))$.

c) Algebraic: as linear operators v on $\mathcal{A}(M)$ satisfying the **Leibnitz rule**:

$$v(fg) = v(f) \cdot g + f \cdot v(g). \tag{7}$$

Such operators are called **derivations** of the algebra $\mathcal{A}(M)$.

We denote the space of all smooth vector fields on M by *Vect*(M). It is a module over the algebra $\mathcal{A}(M)$ in an obvious way.

Exercise 2. Check if this is really obvious for you. Describe explicitly the module structure in the analytic and algebraic interpretations. ♣

The second important example of a vector bundle is the so-called cotangent bundle T^*M, which we now discuss.

A **differential 1-form**, or a **covector field** on M, is defined in two ways:

a) Analytically, as an expression $\sum_{i=1}^n b_i(x)dx^i$ in each local coordinate system on M with the natural transformation rule under the coordinate changing. Namely, in terms of other local coordinates $\{y^j,\ 1 \leq j \leq n\}$ our 1-form looks like $\sum_j \tilde{b}_j(y)dy^j$ where $\tilde{b}_j(y) = \sum_i b_i(x(y))\frac{\partial x^i}{\partial y^j}$.

b) Algebraically, as an $\mathcal{A}(M)$-linear map from *Vect*(M) to $\mathcal{A}(M)$.

Indeed, any such map has the form $\sum_i a^i(x)\frac{\partial}{\partial x^i} \mapsto \sum_i a^i(x)b_i(x)$, hence corresponds to the form $\sum_{i=1}^n b_i(x)dx^i$.

We denote the space of all smooth covector fields on M by $\Omega^1(M)$. It is also a module over $\mathcal{A}(M)$ that is naturally dual to *Vect*(M). In any local coordinate system $(U, \{x^i\}_{1\leq j\leq n})$ the quantities $\{\partial_i := \frac{\partial}{\partial x^i}\}_{1\leq i\leq n}$ and $\{dx^i\}_{1\leq i\leq n}$ form dual bases in the $\mathcal{A}(U)$-modules *Vect*(U) and $\Omega^1(U)$, respectively.

We can interpret a local expression $\omega = \omega_i dx^i$ of a covector field as a section of some vector bundle $\mathbb{R}^n \longrightarrow T^*M \xrightarrow{p} M$. The fiber of this bundle over a point $m \in M$ is naturally identified with the dual vector space to T_mM. This is why T^*M is called the **cotangent bundle** to M.

The space $\Omega^k(M)$ of smooth **differential k-forms** on M can be defined as the k-th exterior power of the $\mathcal{A}(M)$-module $\Omega^1(M)$. In a local coordinate system a k-form ω looks like

$$\omega = \sum_{i_1,\dots,i_k} \omega_{i_1,\dots,i_k}(x)dx^{i_1} \wedge \dots \wedge dx^{i_k} \tag{8}$$

where all indices i_s, $1 \le s \le k$, run from 1 to n and the wedge product $\wedge$ is bilinear, associative, and antisymmetric.

The geometric meaning of differential forms is related to integration over manifolds and will be discussed later.

We also define the space $Vect^k(M)$ of smooth **polyvector fields** on M as the k-th exterior power of the $\mathcal{A}(M)$-module $Vect(M)$. Locally, a k-vector field c is given by an expression

$$c = \sum_{i_1,\dots,i_k} c^{i_1,\dots,i_k}(x)\partial_{i_1} \wedge \dots \wedge \partial_{i_k}. \tag{9}$$

Both $\Omega^k(M)$ and $Vect^k(M)$ can be interpreted as spaces of smooth sections of vector bundles $\wedge^k T(M)$ and $\wedge^k T^*(M)$, respectively.

Remark 5. We shall usually follow **Einstein's rule** and omit the summation sign in expressions like (8) or (9) where a letter occurs twice as a lower and an upper index. For example, we write a vector field v as $v^i\partial_i$ and a differential 1-form ω as $\omega_i dx^i$. The value of ω on v will be the function $\omega(v) = \omega_i v^i$.

This notation not only saves space but sometimes suggests the right formulation of a theorem or a solution to a problem. Consider, for example, Newton's Second Law, which relates the force $\vec{f}$, the mass m, and the acceleration $\vec{a}$:

$$\vec{f} = m \cdot \vec{a}. \tag{10}$$

If we take into consideration that geometrically the force is a covector field (i.e. a differential 1-form) and the acceleration is a vector field (the time derivative of the velocity), we see that (10) is a "wrong" equation: the left-hand side has the lower index while the right-hand side has the upper one.

A way to correct equation (10) is to replace the scalar m by the tensor field[5] m_{ij} with two lower indices and rewrite it in the form

$$f_i = m_{ij}a^j. \tag{10'}$$

The most common geometric example of a tensor with two lower indices is the Riemann metric tensor g_{ij}. We conclude that the mass must be related to the metric, which is the main principle of General Relativity! $\heartsuit$

We finish this section with a pure algebraic definition of a vector bundle.

[5]See the next section for the definition of tensor fields.

Lemma 2. *Let M be a compact manifold, and let $L \xrightarrow{V} M$ be a vector bundle over M with a finite-dimensional fiber V. Then the space $\Gamma(L, M)$ of all smooth sections of L is finitely generated as an $\mathcal{A}(M)$-module.*

Proof. For any point $m \in M$ we can find a neighborhood U such that L is trivial over U. Then $\Gamma(L, U) \simeq C^\infty(U, V)$ is a free module of rank $n = \dim V$ over $C^\infty(U)$ with the basis $v_1, \ldots, v_n$. Let φ be a function in $\mathcal{A}(M)$ with $\operatorname{supp} \varphi \subset U$ and $\varphi(m) \neq 0$. The sections $s_i = \varphi \cdot v_i$, $1 \leq i \leq n$, of L over U can be extended by zero to the whole M and take linearly independent values in any point of a smaller neighborhood $U' \subset U$ where $\varphi \neq 0$.

Since M is compact, we can choose finitely many, say N, such neighborhoods that cover all of M. So, we construct nN sections whose values at any point $m \in M$ span the whole fiber V_m. It follows that they generate $\Gamma(L, M)$ as an $\mathcal{A}(M)$-module. □

The lemma implies that the $\mathcal{A}(M)$-module $\Gamma(L, M)$ is a submodule of a free module of finite rank. (Geometrically, it means that our vector bundle L can be considered as a subbundle of a trivial bundle $\widetilde{L}$ with a finite-dimensional fiber $\widetilde{V}$.)

Actually, it is not only a submodule, but a direct summand of $\widetilde{L}$. Such modules are called **projective**. So, we get the algebraic definition of a vector bundle L: the space $\Gamma(L, M)$ is just a projective module over $\mathcal{A}(M)$.

2.2. Geometric objects on manifolds.

We start with a general and somewhat abstract definition. Soon it will be clear that actually it is quite a working approach. Consider the category $\mathcal{C}$ whose objects are smooth manifolds and whose morphisms are open embeddings.

A **natural vector bundle** is a functor L from $\mathcal{C}$ to the category of vector bundles such that $L(M)$ is a vector bundle over M.

Elements of natural vector bundles are called **geometric objects** and a section of a natural vector bundle $L(M)$ is called a **field of geometric objects** on M.

Vectors and covectors are examples of geometric objects while polyvector fields and differential forms are examples of fields of geometric objects.

It follows from the definition that a canonical action of the group $\operatorname{Diff}(M)$ of all diffeomorphisms of M is defined on sections of $L(M)$. More simply, there exists a canonical way to change variables in the local expression for a given geometric object.

For the examples above the algebraic approach allows us to define the action of $\mathrm{Diff}(M)$ in a uniform way. Namely, the group $\mathrm{Diff}(M)$ acts on the algebra $\mathcal{A}(M)$ by algebra automorphisms:

$$\phi^*(f) = f \circ \phi \quad \text{for } \phi \in \mathrm{Diff}(M),\ f \in \mathcal{A}(M). \tag{11}$$

Warning. Actually, (11) is a *right* action and should be written as $f \cdot \phi$. But we prefer to keep the traditional notation $\phi^*(f)$. Note, however, that the confusion between left and right actions may lead to a discrepancy in notation. For example, in some books the definition of Lie brackets of two vector fields has the sign opposite to ours. ♠

All other transformation laws follow from (11) because our geometric objects are defined in terms of the algebra $\mathcal{A}(M)$. Recall that we defined $Vect(M)$ as the set of all derivations of $\mathcal{A}(M)$ (with the canonical $\mathcal{A}(M)$-module structure), $\Omega^1(M)$ as a dual $\mathcal{A}(M)$-module to $Vect(M)$, and $\Omega^k(M)$ and $Vect^k(M)$ as corresponding exterior powers. So, any automorphism of $\mathcal{A}(M)$ automatically defines automorphisms of all these objects.

There are more general geometric objects called tensors. Analytically, we define a general **tensor field of type** $(k,\ l)$ on M as an object T, which is locally given by coefficients $t^{j_1\cdots j_l}_{i_1\cdots i_k}(x)$ with the following transformation law:

$$t^{j_1\cdots j_l}_{i_1\cdots i_k}(y) = \sum_{p_1\ldots p_l} t^{q_1\cdots q_l}_{p_1\cdots p_k}(x(y)) \frac{\partial x^{p_1}}{\partial y^{i_1}} \cdots \frac{\partial x^{p_k}}{\partial y^{i_k}} \cdot \frac{\partial y^{j_1}}{\partial x^{q_1}} \cdots \frac{\partial y^{j_l}}{\partial x^{q_l}}. \tag{12}$$

To remember this rather involved rule, just keep in mind that from the coefficients of a tensor using the Einstein rule we can construct the expression

$$T = t^{j_1\cdots j_l}_{i_1\cdots i_k}(x) dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_l}},$$

which has an invariant meaning independent of the local coordinate system.

Algebraically, the set $\mathcal{T}^{(k,\ l)}(M)$ of all tensor fields of type $(k,\ l)$ on M is the tensor product over $\mathcal{A}(M)$ of k copies of $\Omega^1(M)$ and l copies of $Vect(M)$.

It can also be defined as the $\mathcal{A}(M)$-module

$$\mathrm{Hom}_{\mathcal{A}(M)}(\underbrace{Vect(M) \times \cdots \times Vect(M)}_{k \text{ times}} \times \underbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}_{l \text{ times}}, \mathcal{A}(M))$$

of all maps that are multilinear over $\mathcal{A}(M)$.

Observe that k-vector fields and differential k-forms are particular cases of tensor fields (antisymmetric tensor fields of types $(0,\ k)$ and $(k,\ 0)$, respectively).

All tensor fields possess the following properties:

1) They are transformed linearly under the change of variables.

2) In their transformation law only the first derivatives of the new variables occur with respect to the old ones or vice versa.

Therefore tensor fields are called **linear geometric objects of first order**. Actually, they are almost all objects of this type.

More general linear geometric objects of first order are the so-called tensor densities (of two kinds) of degree $\lambda \in \mathbb{C}$. The precise definition is as follows.

A **tensor density of the first kind** that has type $(k,\ l)$ and degree λ is a geometric object with the same type of coordinates as a tensor field of type (k, l). But the transformation law of these coordinates, as compared to (12), contains the additional factor: the λ-th power of the absolute value of the Jacobian $J := \det \| \frac{\partial x^i}{\partial y^j} \|$.

The definition of a **tensor density of the second kind** is obtained by adding one more factor in the transformation law: the sign of the Jacobian.

If we consider the category of oriented manifolds with orientation preserving diffeomorphisms, then there is no difference between tensor densities of first and second kind.

We discuss now in some detail the classification problem for linear geometric objects of first order. For technical reasons it is convenient to start with connected oriented manifolds.

It is known that for any connected manifold M the group $G = \mathrm{Diff}(M)$ acts transitively on M. So, any natural vector bundle $L = L(M)$ is a homogeneous G-bundle. In Appendix III we show that any homogeneous G-bundle is determined by the action of the subgroup $H = Stab(m) \subset G$ on the fiber over a point $m \in M$. For geometric objects of first order this problem is reduced to the description of all finite-dimensional linear representations of the connected Lie group $GL_+(n,\ \mathbb{R}) = \{g \in GL(n,\ \mathbb{R}) \mid \det g > 0\}$. The most interesting and important objects are related to irreducible representations. So, for the reader's convenience, we collect the necessary information here.

Here the following notation and terminology are used:

$G = GL_+(n,\ \mathbb{R})$,

H – the abelian subgroup of diagonal matrices in G;

$(\pi,\ V)$ – a finite-dimensional complex irreducible representation of G;

weight vector – a vector $v \in V$ that is a common eigenvector for all operators $\pi(h)$, $h \in H$;

weight – a vector $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$ satisfying the condition $\mu_k - \mu_j \in Z$ for all k, j;

dominant weight – a weight satisfying the additional condition

$$\mu_k - \mu_j \geq 0 \quad \text{for all } k \geq j; \tag{13}$$

we say that a weight μ is bigger than ν if $\mu - \nu$ is a dominant weight;

$e^\mu(h)$ or, simply, h^μ – the homomorphism of H to $\mathbb{C}^\times$ given by

$$e^\mu(h) = \prod_{k=1}^{n} h_k^{\mu_k - \mu_n} \cdot (\det h)^{\mu_n} \quad \text{for } h = \operatorname{diag}(h_1, \ldots, h_n) \in H; \tag{14}$$

a weight vector $v \in V$ is said to be of weight μ if $\pi(h)v = h^\mu v$ for all $h \in H$.

Proposition 4. *Let (π, V) be a finite-dimensional complex irreducible representation of the group $G = GL_+(n, \mathbb{R})$. Then*

a) *the space V has a basis consisting of weight vectors; the corresponding weights form the multiset $Wt(\pi)$ of* **weights of** π*; the multiplicity of a weight μ is denoted by $m_\pi(\mu)$;*

b) *among the weights of π there is exactly one maximal weight; it has multiplicity 1 and is called the* **highest weight***; the corresponding weight vector $v \in V$ is characterized by the property of being invariant under all operators $\pi(g)$ where g is a strictly upper triangular matrix:*

$$g = \begin{pmatrix} 1 & g_{12} & \ldots & g_{1n} \\ 0 & 1 & \ldots & g_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{pmatrix};$$

c) *equivalent representations have the same highest weight; the representation (π^*, V^*), dual to (π, V), has the highest weight*

$$\lambda^* = (-\lambda_n, \ldots, -\lambda_1); \tag{15}$$

d) *for any dominant weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ there is exactly one (up to equivalence) irreducible representation π_λ with the highest weight λ;*

e) *any real N-dimensional irreducible representation (π_0, V_0) of G produces a complex representation (π, V) with $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$; the representation (π, V) is either irreducible with a real highest weight λ (then the initial representation (π_0, V_0) is called* **absolutely irreducible***), or a sum of two irreducible representations with complex-conjugated highest weights λ and $\overline{\lambda}$.* $\square$

It follows from the above proposition that the equivalence classes of absolutely irreducible real representations of $GL_+(n, \mathbb{R})$ are labelled by real highest weights. It turns out that all geometric objects corresponding to these representations are tensor densities of certain symmetry type with respect to permutations of indices.

We list here the highest weights of representations related to the most important geometric objects:

geometric objects	highest weights
functions	$(0, 0, \ldots, 0, 0)$
vector fields	$(0, 0, \ldots, 0, -1)$
k-vector fields	$(0, 0, \ldots, 0, \underbrace{-1, -1, \ldots, -1}_{k \text{ times}})$
differential k-forms	$(\underbrace{1, 1, \ldots, 1}_{k \text{ times}}, 0, 0, \ldots, 0)$
symmetric tensors of type $(k, 0)$	$(k, 0, \ldots, 0)$
λ-th power of a volume form	$(\lambda, \lambda, \ldots, \lambda)$

The general situation, when we consider non-orientable manifolds and general diffeomorphisms, is related to the representations of the full linear group $GL(n, \mathbb{R})$. Here again the absolutely irreducible representations correspond to tensor densities of the first or second kind.

An example of a geometric object with $\epsilon = 1$ is the so-called axial vector field (magnetic vector field in physics), which changes the sign when the orientation is changed.

2.3.* Natural operations on geometric objects.

A powerful tool used to study geometric objects is the **natural operations** on these objects, i.e. algebraic and differential operators which commute with the action of all diffeomorphisms. Therefore it would be very useful to know all the natural operations on geometric objects.

But this problem is rather complicated. An even more specialized problem, to describe all polylinear natural operations on tensor fields, is solved only in the simplest cases of linear and bilinear operations.

Let us discuss some properties of natural operations.

Using translation invariance, one can show that in any local coordinate system a polylinear natural operation on tensor fields is given by a polydifferential operator with constant coefficients.

Moreover, if we consider polylinear natural operations on tensor fields of given types $\tau_i = (k_i, l_i)$, then the following Finiteness Theorem holds.

Theorem 3 (see [**Ki8**] or [**KMS**]). *The vector space $\mathcal{L}(\tau_1, \tau_2, \ldots, \tau_m; \tau)$ of all natural polydifferential operators from $\mathcal{T}^{\tau_1} \times \cdots \times \mathcal{T}^{\tau_m}$ to $\mathcal{T}^{\tau}$ is finite dimensional.*

Now we list some known polylinear natural operations.

The only example of a linear natural operation of differential order 1 is the exterior differential d acting on differential forms. There is no invariant linear operation of differential order ≥ 2. (In particular, this implies that $d^2 = 0$.) There are many bilinear natural operations of differential order 1. The full classification was obtained in [**Grz**].

Here we discuss three of the most important examples.

1. The **Lie derivative**. For any vector field $v \in \mathit{Vect}\,(M)$ and any space of geometric objects $\mathcal{T}(M)$ the operator L_v in $\mathcal{T}(M)$ is defined as $L_v = \frac{d}{dt}\Phi^*(t)\,\big|_{t=0}$. Here the diffeomorphism $\Phi(t)$, denoted also by $\exp tv$, is defined as a map $x \mapsto x(t)$ where $x(t)$ is the solution to the equation

$$\dot{x}(t) = v(x(t)), \qquad x(0) = x.$$

(Actually, the solution exists only locally, but it is enough to define the diffeomorphism $\Phi(t)$ in a neighborhood of a given point $x \in M$ for small t. This in turn is enough to define L_v.)

Note the three particular cases of the Lie derivative:

a) the ordinary derivative vf of a function f along the vector field v is just $L_v f$;

b) the so-called **Lie bracket** of two vector fields is $[v, w] = L_v w = -L_w v$;

c) the Lie derivative L_v defines a derivation of degree zero of the graded algebra $\Omega(M) = \bigoplus_{k=0}^{\dim M} \Omega^k(M)$. There are two other derivations: the exterior differential d of degree $+1$ and the interior multiplication i_v by a vector field v of degree -1. These three derivations are related by the

Cartan Formula.

$$L_v = [d, i_v] := d \circ i_v + i_v \circ d. \tag{16}$$

Proof. The right-hand side is exactly a **supercommutator** (see item 3 below for the definition and properties of "superobjects") of two odd derivations d and i_v. Therefore, it is an even derivation. Since both sides are derivations of $\Omega(M)$, it is enough to check (16) on generators, i.e. on forms of type $f \in \Omega^0(M)$ and $df \in \Omega^1(M)$.

Using the relations $i_v f = 0$ and $d^2 = 0$ we obtain

$$L_v \circ d = d \circ L_v, \quad L_v f = vf = (i_v \circ d + d \circ i_v) f,$$
$$L_v df = dL_v f = (d \circ i_v + i_v \circ d) df.$$

□

A very useful property of the Lie derivative is the

Generalized Leibnitz rule. *If F is any k-linear natural operation on geometric objects, then*

$$L_v F(A_1, \dots, A_k) = \sum_{i=1}^{k} F(A_1, \dots, L_v A_i, \dots, A_k). \tag{17}$$

Example 8. Consider the differentiation of an object A along the vector field v as a bilinear operation $F(v, A) = L_v A$. Then the generalized Leibnitz rule gives:

$$L_w F(v, A) = F(L_w v, A) + F(v, L_w A) \quad \text{or} \quad L_{[w,v]} = L_w L_v - L_v L_w.$$

So, the Lie derivative defines the action of the Lie algebra $Vect(M)$ on the space of geometric objects of a given kind. ♢

Exercise 3. Using the generalized Leibnitz rule, derive the following formula for the exterior differential of a k-form ω:

$$\begin{aligned} d\omega(v_0, \dots, v_k) &= \sum_{i=0}^{k} (-1)^i v_i \omega(v_0, \dots, \widehat{v_i}, \dots, v_k) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([v_i, v_j], v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_k). \end{aligned} \tag{18}$$

For future use we write the important particular cases $k = 0, 1, 2$ separately:

$$\begin{aligned} df(\xi) &= \xi f, \qquad d\theta(\xi, \eta) = \xi\theta(\eta) - \eta\theta(\xi) - \theta([\xi, \eta]), \\ d\omega(\xi, \eta, \zeta) &= \circlearrowleft \xi\omega(\eta, \zeta) - \circlearrowleft \omega([\xi, \eta], \zeta) \end{aligned} \tag{19}$$

where the sign $\circlearrowleft$ denotes the sum over cyclic permutations of ξ, η, ζ. ♣

2. The **Schouten bracket** is a bilinear operation on polyvector fields on a manifold. It defines on the graded space of polyvector fields the structure of a so-called **graded Lie algebra**, or **Lie superalgebra**, which extends the structure of a Lie algebra on $Vect(M)$.

Proposition 5. *There exists a unique bilinear operation* $[\ ,\ ]$ *on the space of polyvector fields which has the properties:*

a) *For any* $v \in Vect^k(M)$, $w \in Vect^l(M)$ *we have*
$[u, v] \in Vect^{k+l-1}(M)$ (*Homogeneity*)

b) $[u, v \wedge w] = [u, v] \wedge w + (-1)^k v \wedge [u, w]$ (*Distributive Law*)

c) *For* $k = l = 1$ *the operation is the ordinary Lie bracket of vector fields.*

Moreover, this operation is natural (i.e. commutes with all diffeomorphisms) and satisfies the identities:

$[u, v] = -(-1)^{(k-1)(l-1)}[v, u]$ (*Super antisymmetry*)

$[u, [v, w]] = [[u, v], w] + (-1)^{(k-1)(l-1)}[v, [u, w]]$ (*Super Jacobi identity*).

Proof. The uniqueness follows immediately from properties a), b), and c).

Indeed, these properties imply that for decomposable polyvector fields v and w the operation is defined by the formula

$$(20)\quad [v_1 \wedge \cdots \wedge v_k,\ w_1 \wedge \cdots \wedge w_l] = \sum_{i,j} (-1)^{i+j+k-1}[v_i,\ w_j] \wedge v_1 \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge \widehat{w_j} \wedge \cdots \wedge w_l.$$

We could use (20) as a definition of our operation, but we need to check that it is correct, i.e. does not depend on the way a polyvector field as a sum of wedge products of vector fields is written.

For this end one can use an alternative definition of the Schouten bracket. Fix a volume form *vol* on our manifold and define a map $\delta : Vect^k(M) \to Vect^{k-1}(M)$ by the formula

$$(21)\qquad \delta(v^{i_1\cdots i_k}\partial_{i_1} \wedge \cdots \wedge \partial_{i_k}) = \partial_{i_1} v^{i_1\cdots i_k}\partial_{i_2} \wedge \cdots \wedge \partial_{i_k}$$

in the special local coordinate system where $vol = dx^1 \wedge \cdots \wedge dx^n$. (Such coordinate systems are called **unimodular**.) For $k = 1$ the map $\delta : Vect(M) \longrightarrow C^\infty(M)$ is the usual **divergence** of a vector field.

The second definition of the Schouten bracket is

$$(22)\quad [v,\ w] = \delta(v \wedge w) - \delta(v) \wedge w - (-1)^k v \wedge \delta(w) \quad \text{for } v \in Vect^k(M).$$

This definition formally depends on the choice of a volume form *vol* but actually it does not.

Exercise 4. Show that the right-hand side of (22) does not change if the volume form *vol* is multiplied by a function ρ.

Hint. Check that the map δ' corresponding to the form $vol' = \rho \cdot vol$ is related to the initial map δ by the equation

$$\delta'(v) = \delta(v) - \sum_{i=1}^{k} (-1)^i \frac{v_i \rho}{\rho} v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k$$

where the hat means that the factor must be omitted. ♣

It remains to check that (20) and (22) define the same operation and this operation has the announced properties. We leave this to the reader.□

Remark 6. Observe that the homogeneity property of the Schouten bracket is related to the grading in the space of polyvector fields in a non-standard way.

To get the standard homogeneity property, we have to assign the grading $k-1$ to k-vector fields.

It is also tempting to extend the bracket operation to a bigger space by including $\Omega^k(M)$ as a homogeneous component of the grading $-1-k$. ♣

3. The **Nijenhuis bracket** is defined for so-called **vector differential forms**, i.e. the elements of $Vect(M) \otimes_{C^\infty(M)} \Omega(M)$. The *raison d'être* of this operation is the algebraic structure of $\Omega(M)$ as an associative supercommutative differential algebra.

Recall that a $\mathbb{Z}$-graded associative algebra $\mathcal{A}$ is called **supercommutative** if for homogeneous elements $a, b \in \mathcal{A}$ we have $ab = (-1)^{\alpha\beta} ba$ where $\alpha = \deg a,\ \beta = \deg b$.

A **derivation of degree** k of a commutative superalgebra $\mathcal{A}$ is a linear map $D : \mathcal{A} \to \mathcal{A}$ which sends $\mathcal{A}^m$ to $\mathcal{A}^{m+k}$ and satisfies the **super Leibnitz rule**:

$$D(ab) = D(a)b + (-1)^{k\alpha} aD(b), \qquad \alpha = \deg a.$$

We now return to differential forms. Put $\mathcal{A} = \Omega(M)$ and observe that this is a commutative superalgebra with the derivation d of degree $+1$.

We shall consider only those derivations D of $\Omega(M)$ which supercommute with d:

$$D(d\omega) = (-1)^{\deg D} d(D\omega).$$

From this property and the Leibnitz rule we see that D is completely determined by its restriction to $\Omega^0(M) = C^\infty(M)$. The image of this restriction is contained in $\Omega^{\deg D}(M)$. Moreover, from the Leibnitz rule applied to the case of the product of two functions, we get

$$Df = \omega^i \partial_i f \quad \text{where } \omega^i \in \Omega^{\deg D}(M),\ 1 \le i \le n. \tag{23}$$

One can verify that any collection of k-forms $\{\omega^i\}_{1\le i\le n}$ gives rise to a derivation of degree k whose restriction to $\Omega^0(M)$ is given by (23). So, these derivations can be identified with geometric objects from $Vect(M)\otimes_{C^\infty(M)}\Omega^k(M)$: $D \longleftrightarrow \partial_i \otimes \omega^i$.

The derivations of a graded superalgebra form themselves an algebra with respect to the **supercommutator**

$$[D_1,\ D_2] = D_1 \circ D_2 - (-1)^{\deg D_1\cdot\deg D_2} D_2 \circ D_1. \tag{24}$$

The right-hand side of (24) is a derivation of degree $\deg D_1 + \deg D_2$ and the supercommutator satisfies the **super Jacobi identity**:

$$[D_1,\ [D_2,\ D_3]] = [[D_1,\ D_2],\ D_3] + (-1)^{\deg D_1 \deg D_2}[D_2,\ [D_1,\ D_3]]. \tag{25}$$

In our case this supercommutator is exactly the Nijenhuis bracket.

When $\deg D_1 = 0$ or $\deg D_2 = 0$ we regain the Lie derivative. The simplest new operation arises when $\deg D_1 = \deg D_2 = 1$. The corresponding geometric objects can be interpreted as fields of linear operators in the tangent spaces T_xM. They can also be considered as $\mathcal{A}(M)$-linear operators on $Vect(M)$.

The Nijenhuis bracket of two such operators is a vector 2-form that can be interpreted as a family of bilinear operations ("multiplications") in T_xM or as an $\mathcal{A}(M)$-bilinear operator from $Vect(M) \times Vect(M)$ to $Vect(M)$.

In this case the operation is symmetric: $[A,\ B] = [B,\ A]$. Therefore it is determined by the corresponding quadratic operation: $A \mapsto [A,\ A]$. The latter admits a simple explicit formula:

$$[A,\ A](\xi,\ \eta) = [A\xi,\ A\eta] - A[\xi,\ A\eta] - A[A\xi,\ \eta] + A^2[\xi,\ \eta]. \tag{26}$$

It shows that the resulting bilinear operator $[A,\ A]$ is antisymmetric in $\xi,\ \eta$.

Exercise 5. Check directly that the right-hand side of (24) is bilinear in ξ and η not only over $\mathbb{R}$ but also over $\mathcal{A}(M)$.

Hint. Use the formula $[f\xi,\ \eta] = f[\xi,\ \eta] - \eta f \cdot \xi$, which follows from the generalized Leibniz rule. ♣

Many important geometric theorems use the just described case of the Nijenhuis bracket in their formulations. We mention two examples here.

a) Recall that an **almost complex structure** on M is defined by a tensor field $J \in T^{1,1}$ satisfying

$$J^i_j J^j_k = -\delta^i_k \qquad \text{(or simply } J^2 = -1\text{)}.$$

It introduces a complex structure on every tangent space T_xM.

Clearly, each complex manifold M possesses an almost complex structure but the converse is true only under an additional integrability condition.

b) The so-called **almost product structure** on M is given by a tensor field $P \in T^{1,1}$ satisfying

$$P^i_j P^j_k = P^i_k \qquad (\text{or simply } P^2 = P).$$

It defines on every tangent space T_xM the structure of a direct sum: $T_xM = P(T_xM) \oplus (1 - P)(T_xM)$. This structure arises each time M is a product of two manifolds: $M = M_1 \times M_2$. Here again, the converse is true (even locally) only under an additional **integrability condition**.

Proposition 6. *The integrability condition for an almost complex structure J (respectively for an almost product structure P) has the form*

$$[J,\, J] = 0 \qquad (\textit{resp. } [P,\, P] = 0). \tag{27}$$

Proof. The necessity of (24) is easy: if J comes from a complex structure (respectively P comes from a product structure), then in appropriate coordinates this tensor has constant coefficients. Hence, $[J,\, J] = 0$ (resp. $[P,\, P] = 0$).

Conversely, let $[J,\, J] = 0$ (resp. $[P,\, P] = 0$). Consider the complex geometric distribution $A = \ker(J - i1)$ (resp. the real geometric distribution $B = \ker(P - 1)$). From (26) we derive that the space $Vect(M,\, A)$ of admissible vector fields is a Lie subalgebra in $Vect^{\mathbb{C}}(M)$ (resp. $Vect(M,\, B)$ is a Lie subalgebra in $Vect(M)$). Hence, the distribution satisfies the Frobenius criterion (see Section 3.4 below). □

2.4. Integration on manifolds.

Here we recall the basic facts about **integration** of differential forms and densities over manifolds.

In calculus courses you are taught to integrate functions. Actually, this is not quite correct. The object you are dealing with is not a function integrated over a set, but a differential form of top degree integrated over an oriented manifold or a density of the first kind and degree 1 integrated over an arbitrary manifold.

This is stressed by the notation suggested by Leibnitz 300 years ago (and survived to the present time!):

$$\int_a^b f(x)dx \qquad \text{or} \qquad \int_{\mathbb{R}} f(x)|dx|.$$

In the first case we have an integral of a 1-form $f(x)dx$ over the oriented segment $[a, b]$. In the second case we integrate a density $f(x)|dx|$ over a non-oriented real line.

Both types of integration have their advantages and disadvantages.

The first one relates the analytic notion of the exterior differential of a differential form with the geometric notion of the boundary of a smooth manifold. The famous **Stokes formula**,

$$\int_M d\omega = \int_{\partial M} \omega, \tag{28}$$

is a very powerful method both to compute integrals and to study the geometry of smooth manifolds.

In this formula we assume that the orientations of M and ∂M are related in a definite way. Namely, if a neighborhood of a boundary point of M is covered by a positive chart with coordinates $(x^1, \ldots, x^n)$, satisfying the inequality $x^n < 0$ for interior points and the equality $x^n = 0$ for boundary points, then the coordinates $(x^1, \ldots, x^{n-1})$ form a positive chart for ∂M. This definition does not work for $n = 1$ and needs to be slightly modified. We leave it to the reader.

The second one has a natural generalization where a smooth manifold M is replaced by an arbitrary set X and the density $|dx|$ is replaced by a measure μ defined on some collection of subsets in X. In most applications X is assumed to be a locally compact topological space and μ a Borel measure defined on all Borel subsets.[6]

Remark 7. The expressions $\int_X f(x)$ and $\int f(x)$, often occurring in student's notes, do not make any sense. Try, for example, to define what the integral of the function $f \equiv 1$ over the edge of the table is. The possible answer: "the length of the table" leads immediately to the next question: in which units is the length measured?

The missing part dx just shows the scale, introducing a parametrization of the geometric object (e.g. the edge of the table) by a segment of the real line. ♣

Now we give the general definition of the integral of the first kind. Let M be a smooth oriented manifold and ω a differential form of the top degree $m = \dim M$. We assume that ω has compact support (i.e. vanishes outside some compact subset $K \subset M$). Our goal is to define the integral $\int_M \omega$.

[6]Recall that **Borel subsets** form a minimal collection of subsets in X that contains all open sets and is closed under set-theoretic operations: countable union, countable intersection, and taking the complement.

First, we reduce the general problem to the simple case when our manifold is a bounded domain in $\mathbb{R}^n$. To this end we choose a covering of our manifold M by local charts $\{U_\alpha\}_{\alpha\in A}$ with the following properties:

1. all U_α are pairwise positively related and define the orientation of M;
2. all U_α have compact closures in M;
3. any compact subset $K \subset M$ intersects only finitely many U_α.

We do not prove the existence of such a covering here. In all practical situations it is rather obvious.

Further, it is known that in this situation there exists a system of functions $\{\phi_\alpha\}_{\alpha\in A}$ such that

$$\phi_\alpha \in \mathcal{A}(M), \qquad \phi_\alpha \geq 0, \qquad \text{supp } \phi_\alpha \subset U_\alpha, \qquad \sum_{\alpha\in A} \phi_\alpha = 1.$$

This system of functions is called a **partition of unity** subordinated to the given covering $\{U_\alpha\}_{\alpha\in A}$.

We define the integral in question as a sum over $\alpha \in A$ of the integrals

$$I_\alpha = \int_{U_\alpha} \phi_\alpha \cdot \omega.$$

Note that actually only finitely many summands are different from zero.

Using local coordinates $(x_1, \ldots, x_m)$ in U_α, we write $\phi_\alpha \cdot \omega$ as $f(x)d^m x$,[7] and define I_α as the integral of a smooth function $f(x)$ over a domain $V_\alpha \subset \mathbb{R}^n$. The latter integral is defined as the limit of Riemann integral sums. Namely, we split V_α into small pieces V_i, choose a point $x_i \in V_i$, and introduce the sum

$$S(f; \{V_i\}, \{x_i\}) \;=\; \sum_i f(x_i) \cdot vol(V_i).$$

Since the integrand is continuous and has compact support, the integral sums indeed have a limit when the maximal diameter of the parts $\{V_i\}$ tends to zero.

It remains to check that the sum $\sum_{\alpha\in A} I_\alpha$ does not depend on the choice of a covering $\{U_\alpha\}$ and a partition $\{\phi_\alpha\}$. Let $\{U'_\beta\}_{\beta\in B}$ be another covering with the same properties, and let $\{\psi_\beta\}_{\beta\in B}$ be the partition of unity subordinated to it.

Lemma 3. *Put* $I'_\beta = \int_{U'_\beta} \psi_\beta \cdot \omega$. *Then*

$$\sum_{\alpha\in A} I_\alpha = \sum_{\beta\in B} I'_\beta.$$

[7]Here $d^m x$ is a short expression for $dx^1 \wedge \cdots \wedge dx^m$.

Proof. Consider the third covering $\{U''_\gamma\}_{\gamma\in\Gamma}$, which is finer than either of the previous ones. It is convenient to put $\Gamma = A\times B$ and $U''_\gamma = U_\alpha \cap U'_\beta$ for $\gamma = (\alpha,\, \beta)$. Let $\chi_\gamma = \phi_\alpha \psi_\beta$ for $\gamma = (\alpha,\, \beta)$. Then $\{\chi_\gamma\}_{\gamma\in\Gamma}$ is the partition of unity subordinated to $\{U''_\gamma\}_{\gamma\in\Gamma}$. Put $I''_\gamma = \int_{U''_\gamma} \chi_\gamma \cdot \omega$.

We live it to the reader to check the equalities:

$$I_\alpha = \sum_{\beta\in B} I''_{(\alpha,\beta)}, \qquad I'_\beta = \sum_{\alpha\in A} I''_{(\alpha,\beta)}, \qquad I := \sum_{\gamma\in\Gamma} I''_\gamma = \sum_{\alpha\in A} I_\alpha = \sum_{\beta\in B} I'_\beta.$$

□

3. Symplectic and Poisson manifolds

3.1. Symplectic manifolds.

By definition, a **symplectic manifold** is a pair $(M,\, \sigma)$ where M is a smooth manifold and $\sigma \in \Omega^2(M)$ is a non-degenerate closed differential 2-form on M. In a local coordinate system σ has the form $\sigma_{i,j}(x)dx^i \wedge dx^j$ where $\sigma = \|\sigma_{i,j}(x)\|$ is a skew-symmetric non-degenerate matrix. Since such matrices exist only in even-dimensional vector spaces, all symplectic manifolds are even-dimensional.

Darboux Theorem. *Any non-degenerate closed differential 2-form in an appropriate local coordinate system $(p_1,\, \dots,\, p_n,\, q^1,\, \dots,\, q^n)$ has the following canonical form:*

$$\sigma = dp_i \wedge dq^i. \tag{29}$$

□

This theorem shows that the notion of symplectic manifold is an odd analogue of the notion of a Riemannian manifold with a flat metric.

The coordinates $(p_i,\, q^j)$ above are called **canonical coordinates**. They are defined up to so-called **canonical transformation**, i.e. diffeomorphism, preserving the basic form σ. Such transformations are also called **symplectomorphisms**.

We call a vector field v **symplectic** if it defines an infinitesimal automorphism of $(M,\, \sigma)$, i.e. if $L_v\sigma = 0$. In other words, the flow generated on M by a symplectic vector field consists of canonical transformations of M.

We can use the basic form σ to raise and lower the indices of tensor fields, exactly as in the Riemannian case. In particular, we can establish a bijection between $Vect\,(M)$ and $\Omega^1(M)$ (also denoted by σ):

$$v = v^i\partial_i \xrightarrow{\sigma} \theta = \theta_k dx^k \qquad \text{where } \theta_k = \sigma_{k,i}v^i \text{ or } \theta = -i_v\sigma. \tag{30}$$

The **skew-gradient** of a function $f \in C^\infty(M)$ is a vector field s-grad $f := \sigma^{-1}df$, corresponding to the 1-form df (i.e. $i_v\sigma = -df$). In canonical coordinates we have

$$\text{s-grad}\, f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{31}$$

In particular,

$$\text{s-grad}\, p_i = \frac{\partial}{\partial q^i}, \qquad \text{s-grad}\, q^i = -\frac{\partial}{\partial p_i}.$$

Exercise 6. Let S be a smooth oriented 2-dimensional surface with a metric g_{ij} and a symplectic form ω_{ij}. Assume that both structures define the same volume form on S, i.e. $\det \|g_{ij}\| = \det \|\omega_{ij}\|$. Show that the skew-gradient s-grad f is obtained from the ordinary gradient grad f by a rotation on the right angle.

Hint. In appropriate local coordinates at a given point we have

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \omega_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

♣

Usually, skew-gradient vector fields are called **Hamiltonian fields** and the function f is called the **generating function**, or simply the **generator** of s-grad f. We denote the collection of all symplectic vector fields on M by $Vect(M, \sigma)$ and the collection of Hamiltonian vector fields by $Ham(M, \sigma)$.

Theorem 4. a) *All Hamiltonian vector fields are symplectic.*

b) *If the manifold M is simply connected, the converse is also true.*

c) *The commutator $[v, w]$ of two symplectic vector fields is a Hamiltonian vector field with the generator $\sigma(v, w)$.*

Proof. a) Let us recall the Cartan formula (see formula (16) in Section 2.3):

$$L_v = d \circ i_v + i_v \circ d.$$

Applying it to the form σ, we get $L_v\sigma = d\theta$ where $\theta = i_v\sigma = -\sigma^{-1}v$. If $v =$ s-grad f is a Hamiltonian field, we have $\theta = -df$ and $L_v\sigma = -d^2 f = 0$. So, v is symplectic.

b) If v is symplectic, we get $d\theta = 0$, i.e. the form θ is closed. On a simply connected manifold every closed form is exact. Hence, $\theta = df$ for some smooth function f and $v =$ s-grad f is a Hamiltonian vector field.

c) Let v, w be symplectic vector fields and $f = \sigma(v, w)$. Then $f = (i_w \circ i_v)\sigma$. Therefore $df = (d \circ i_w \circ i_v)\sigma = (L_w - i_w \circ d)i_v\sigma$. But $i_v\sigma$ is closed,

hence $df = L_w(i_v\sigma) = i_{L_w v}\sigma = -i_{[v,w]}\sigma$, i.e. $[v, w] =$ s-grad f. (Here we used the generalized Leibniz rule (17) from Section 2.3.) □

A submanifold S of a symplectic manifold (M, σ) is called **isotropic** if $\sigma\,\big|_S = 0$. It is clear that for any isotropic submanifold S we have $\dim S \leq \frac{1}{2}\dim M$. If the equality holds, S is called a **Lagrangian** submanifold.

In representation theory and in geometric quantization an important role is played by so-called **Lagrangian fibrations** $M \xrightarrow{p} B$, i.e. fibrations with Lagrangian fibers.

Let us now turn to the algebraic approach and characterize a manifold M by the algebra $\mathcal{A}(M)$. On any symplectic manifold we have an additional algebraic operation on $\mathcal{A}(M)$ (which is also defined on the bigger space $C^\infty(M)$). It is the so-called **Poisson bracket** $\{\cdot\,,\,\cdot\}$ defined in three equivalent ways:

$$\{f_1,\, f_2\} = (\text{s-grad} f_1) f_2 = -(\text{s-grad} f_2) f_1 = \sigma(\text{s-grad} f_1,\, \text{s-grad} f_2). \tag{32}$$

In any canonical local coordinates we have

$$\{f_1,\, f_2\} = \frac{\partial f_1}{\partial p_i}\frac{\partial f_2}{\partial q^i} - \frac{\partial f_1}{\partial q^i}\frac{\partial f_2}{\partial p_i}. \tag{33}$$

In particular, the following **canonical relations** hold:

$$\{p_i,\, p_j\} = \{q^i,\, q^j\} = 0, \qquad \{p_i,\, q^j\} = \delta_i^j := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{34}$$

Theorem 5. a) *The Poisson bracket defines the structure of an infinite-dimensional Lie algebra on the space $C^\infty(M)$.*

b) *The map $f \mapsto$ s-grad f is a Lie algebra homomorphism from $C^\infty(M)$ to the Lie subalgebra $Ham(M, \sigma) \subset Vect(M, \sigma) \subset Vect(M)$.*

c) *The kernel of the above homomorphism coincides with the center of the Lie algebra $C^\infty(M)$ and consists of all locally constant functions (which are constant on every connected component of M).*

Proof. a) The antisymmetry of the Poisson bracket is evident. We leave it to the reader to check the Jacobi identity (in canonical coordinates it is a simple calculation).

b) This is part c) of Theorem 4.

c) Let f belong to the center of the Lie algebra $C^\infty(M)$. Then s-grad f annihilates all smooth functions, hence must be a zero vector field. Therefore $df = 0$ and f is locally constant. □

The algebraic incarnation of the notion of a Lagrangian fibration is a maximal abelian subalgebra A in the Lie algebra $C^\infty(M)$ of a special kind. Namely, we require that for any $m \in M$ the Lie algebra A contains $n = \frac{1}{2}\dim M$ functions which have independent gradients at m.

Indeed, let $A = p^*\big(C^\infty(B)\big)$. It can be viewed as an algebra of functions on M that are constant along the fibers. Let $P(m) \subset T_mM$ be a tangent space to a fiber of P passing through the point $m \in M$. If a function f is constant along the fibers, then its gradient $df(m)$ is a covector vanishing on $P(m)$. Therefore, its skew-gradient has the property:

$$\text{s-grad}\, f(m) \in P(m)^\perp = P(m).$$

Hence, for any two such functions we have $\{f_1, f_2\} = \sigma(\text{s-grad}\, f_1, \text{s-grad}\, f_2) = 0$.

Conversely, if the family of functions $\{f_\alpha\}_{\alpha\in A}$ has the property $\{f_\alpha, f_\beta\} = 0$ for all $\alpha, \beta \in A$, then their skew-gradients at any point $m \in M$ span an isotropic subspace in $T_m(M)$ of dimension at most $n = \frac{1}{2}\dim M$. Hence, there is at most n functionally independent functions with vanishing Poisson brackets. If $f_1, \ldots, f_n$ is such a family, then the system of equations

$$f_1(x) = c_1, \ \ldots\ , \ f_n(x) = c_n$$

defines a Lagrangian fibration of M.

There are three main sources of symplectic manifolds: cotangent bundles, complex algebraic manifolds, and coadjoint orbits. The last source is considered in detail in the main part of the book. We briefly discuss the other two.

Example 9. Let M be any smooth manifold, and let $X = T^*M$ be the cotangent bundle on M. We can define in a purely geometric way a differential 1-form θ on X. Namely, let $p : X \longrightarrow M$ be the canonical projection. For any $\xi \in T_xX$ we denote by $p_*(\xi)$ the corresponding vector in $T_{p(x)}M$. Now recall that x itself is a covector in $T^*_{p(x)}M$. We define the form θ by

$$\theta(x)(\xi) = \langle x, p_*(\xi)\rangle. \tag{35}$$

For any chart U with local coordinate system $(q^1, \ldots, q^n)$ on M we can define the coordinate system $(q^1, \ldots, q^n; p_1, \ldots, p_n)$ on $T^*(U) \subset T^*(M)$ so that

$$p_i(x) = \langle x, \partial_i\rangle \quad \text{where } \partial_i := \frac{\partial}{\partial q^i}. \tag{36}$$

In these coordinates the form θ looks like

(37) $$\theta = p_i dq^i$$

since $\theta(x)(\frac{\partial}{\partial q^i}) = p_i, \quad \theta(x)(\frac{\partial}{\partial p_j}) = 0$.

Therefore, the 2-form $\omega = d\theta$ is given by the formula (29) and, consequently, is closed and non-degenerate. We see that $T^*(M)$ is indeed a symplectic manifold.

The functions on X that have the form $f = \phi \circ p$ (i.e. functions depending only on q-coordinates) form a maximal abelian subalgebra in $C^\infty(X)$. The corresponding fibration is just the canonical fibration of $X = T^*M$ over M.

There is a convenient description of Lagrangian submanifolds $L \subset X$ for which the projection p is a bijection of L to M. Namely, such a manifold is exactly the range (i.e. set of values) of a closed 1-form on M viewed as a section of the cotangent bundle T^*M.

Indeed, the intersection $L \cap T^*U$ in the coordinate form is given by the equations

$$p_i = a_i(q), \quad 1 \leq i \leq n.$$

The tangent space TL is spanned by vectors $\xi_j = \frac{\partial}{\partial q^j} + \frac{\partial a_k}{\partial q^j}\frac{\partial}{\partial p_k}$. Hence, L is Lagrangian iff $\omega(\xi_i, \xi_j) = 0$ for all i, j, i.e. iff $\frac{\partial a_i}{\partial q^j} - \frac{\partial a_j}{\partial q^i} = 0$. Let θ_L be the associated 1-form on M given by

$$\theta_L(v) = \langle p^{-1}(m), v\rangle \quad \text{for } v \in T_m(M), \qquad \text{or} \qquad \theta_L = a_i(q)dq^i.$$

It is clear that the condition above is equivalent to $d\theta_L = 0$.

Note in conclusion that all symplectic manifolds of the type $X = T^*M$ are non-compact. $\diamondsuit$

Example 10. Complex algebraic geometry provides a lot of examples of compact symplectic manifolds. We start with a reminder of the elements of complex geometry.

Let M be a complex manifold, and let $U \subset M$ be a chart with a local coordinate system $(z^1, \ldots, z^n)$. A **Hermitian form** on M is given locally by the expression

$$h = h_{ij}(z)dz^i \otimes \overline{dz^j}$$

where h_{ij} are complex-valued functions on U satisfying $h_{ji}(z) = \overline{h_{ij}(z)}$.

The real part $g = \Re h$ is a real bilinear form on TU. We keep the same notation for the corresponding quadratic form. Let us introduce the real coordinates $u^k = \Re z^k$, $v^k = \Im z^k$ and the real-valued functions $a_{ij} = a_{ji} = \Re h_{ij}, \quad b_{ij} = -b_{ji} = \Im h_{ij}$. Then we have

$$g = a_{ij}(du^i du^j + dv^i dv^j) + b_{ij}(du^i dv^j + dv^j du^i).$$

The imaginary part of h defines a real differential 2-form ω on M:

$$\omega = a_{ij}(dv^i \wedge du^j) + \frac{1}{2} b_{ij}(du^i \wedge du^j - dv^i \wedge dv^j).$$

A Hermitian form h is called **Kähler** if its real part $g = \Re h$ is positive definite and the imaginary part $\omega = \Im h$ is closed. A complex manifold endowed with a Kähler form is called a **Kähler manifold**.

In this case ω is always non-degenerate, since in an appropriate local coordinate system we have $h_{ij} = \delta_{ij}$, hence $a_{ij} = \delta_{ij}$, $b_{ij} = 0$ and, consequently,

$$g = (du)^2 + (dv)^2, \qquad \omega = dv^i \wedge du^i.$$

So, every Kähler manifold possesses a symplectic structure.

Now, the restriction of a Kähler form to a complex submanifold is again a Kähler form, since both the positivity of $g = \Re h$ and closedness of $\omega = \Im h$ are preserved by a restriction. Therefore, any complex submanifold of a Kähler manifold is itself a Kähler manifold, hence possesses a canonical symplectic structure.

The closedness of $\Im h$ implies that locally there exists a real-valued function K such that

$$h_{ij} = \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^j} K.$$

This function K is called a **Kähler potential** of the form h. It is not unique and is defined modulo a summand of the form $\Re f$ where f is a holomorphic function.

Conversely, if K is any real-valued function, we can define a Hermitian form

$$h = \partial\bar{\partial} K := \left(\frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^j} K(z) \right) \cdot dz^i \otimes \overline{dz^j}.$$

The imaginary part of this form is always closed. If, moreover, the real part is positive definite, we get a Kähler form.[8]

It is well known that the complex projective space $P^N(\mathbb{C})$ has a remarkable Kähler form, the so-called **Fubini–Study form** h, which is uniquely defined by two conditions:

1. The form h is $SU(N+1)$-invariant.
2. For any $P^1(\mathbb{C})$ naturally embedded in $P^N(\mathbb{C})$ we have $\int_{P^1(\mathbb{C})} \omega = 1$.

We give here the explicit expression of this form. Let $(x^0 : x^1 : \ldots : x^N)$ be the homogeneous coordinates in $P^N(\mathbb{C})$. The real functions

$$K_i = \log \frac{\sum_{k=0}^N |x^k|^2}{|x^i|^2}$$

[8]If the real part is non-degenerate, but not positively defined, we obtain a so-called **pseudo-Kähler form**.

have the property

$$K_j - K_i = \log\left(\frac{|x^i|^2}{|x^j|^2}\right) = \log\left(\frac{x^i}{x^j}\right) + \overline{\log\left(\frac{x^i}{x^j}\right)}.$$

We see that the differences $K_j - K_i$ are real parts of analytic functions on $U_i \cap U_j$. Therefore the Hermitian forms $h_i = \partial\overline{\partial} K_i$ and $h_j = \partial\overline{\partial} K_j$ coincide on the intersections $U_i \cap U_j$. Hence, they define a single Hermitian 2-form h on the whole space $P^N(\mathbb{C})$.

In terms of affine coordinates $z = (z^1, \ldots, z^N)$, $z^i = \frac{x^i}{x^0}$, this form looks as follows:

$$h = \partial\overline{\partial} \log(1 + |z|^2) = \frac{|dz|^2}{1+|z|^2} - \left|\frac{z d\overline{z}^t}{1+|z|^2}\right|^2. \tag{38}$$

This form is obviously invariant under the group $SU(N) \subset SU(N+1)$ acting linearly on affine coordinates and also under the group S_{N+1} acting by permutations of $N+1$ homogeneous coordinates. It follows that it is invariant under the group $SU(N+1)$ acting by projective transformations on $P^N(\mathbb{C})$.

The real part of this form is an $SU(N+1)$-invariant metric $g = h_{kj} dz^k \overline{dz^j}$ and the imaginary part is an $SU(N+1)$-invariant symplectic form

$$\omega = \frac{i}{2} \cdot h_{kj} dz^k \wedge \overline{dz^j} \qquad \text{where } h_{kj} = \frac{(1+|z|^2)\delta_{kj} - \overline{z^k} z^j}{(1+|z|^2)^2}. \tag{39}$$

It remains to multiply this form by an appropriate constant in order to satisfy the normalization condition 2 above. We shall do it after some discussion.

The restriction of the Fubini–Study form on any smooth algebraic submanifold $M \subset P^N(\mathbb{C})$ is again a Kähler form whose real part gives a metric on M and the imaginary part defines a symplectic structure on M.

Let n be the complex dimension of M. Then the form

$$vol = \frac{\omega \wedge \cdots \wedge \omega}{n!} \qquad (n \text{ factors})$$

is a volume form on M and the integral of this form over M is an important characteristic of the algebraic manifold M, which is called the **degree** and denoted by $d(M)$. The geometric meaning of the degree can be seen from the following proposition.

Proposition 7. *The intersection of M with a generic projective subspace of the complementary dimension $N - n$ consists of $d(M)$ points.*

In particular, the degree of a smooth hypersurface given by the equation $P(x) = 0$ is equal to d where P is a homogeneous polynomial of degree d. □

The normalization condition means that the degree of $P^1(\mathbb{C}) \subset P^N(\mathbb{C})$ is equal to 1. So we can determine the normalization factor by computing the integral of the form (39) over the submanifold $P^1(\mathbb{C}) \subset P^N(\mathbb{C})$. The result implies that the Fubini-Study form on $P^N(\mathbb{C})$ is equal to

$$h_{FS} = -\frac{1}{2\pi}\partial\bar{\partial}\log(1 + |z|^2). \tag{40}$$

◇

Symplectic manifolds form a category $\mathcal{S}ym$ where the morphisms are symplectomorphisms. This category admits a very interesting extension $\widetilde{\mathcal{S}ym}$. We observe that the graph Γ_φ of a symplectomorphism $\varphi : M_1 \longrightarrow M_2$ is a Lagrangian submanifold in $M_1^\circ \times M_2$. Here M_1° denotes the symplectic manifold $(M_1, -\sigma_1)$ and the symplectic structure on a product $M_1 \times M_2$ is defined as $p_1^*\sigma_1 + p_2^*\sigma_2$.

The objects of $\widetilde{\mathcal{S}ym}$ are again symplectic manifolds, but morphisms from M_1 to M_2 are all Lagrangian submanifolds in $M_1^\circ \times M_2$. The composition of morphisms $f : M_2 \longrightarrow M_3$ and $g : M_1 \longrightarrow M_2$ is defined as a submanifold $\Gamma_{f\circ g} \subset M_1 \times M_3$ that contains all pairs (m_1, m_3) for which there exists a point $m_2 \in M_2$ such that $(m_1, m_2) \in \Gamma_g$, $(m_2, m_3) \in \Gamma_f$.[9] We refer to [**We**] for the applications of the category $\widetilde{\mathcal{S}ym}$.

3.2. Poisson manifolds.

Definition 4. A smooth manifold M is called a **Poisson manifold** if it is endowed with a bivector field $c = c^{ij}\partial_i\partial_j$ such that the **Poisson brackets**

$$\{f_1, f_2\} = c^{ij}\partial_i f_1 \partial_j f_2 \tag{41}$$

define a Lie algebra structure on $C^\infty(M)$.

The Jacobi identity for Poisson brackets imposes on c a non-linear differential condition $[c, c] = 0$ where $[\ ,\ ]$ is the Schouten bracket on polyvector fields (see Section 2.3).

In this particular case the Schouten bracket $[c, c]$ is a trivector t^{ijk} given by

$$t^{ijk}\partial_i f_1 \partial_j f_2 \partial_k f_3 = \circlearrowleft \{f_1, \{f_2, f_3\}\} \tag{42}$$

where $\circlearrowleft$ denotes the summation over the cyclic permutations of f_1, f_2, f_3.

[9] Actually, this composition law is only partially defined. So, $\widetilde{\mathcal{S}ym}$ is not a true category.

Proposition 8. *Any symplectic manifold (M, σ) has a canonical Poisson structure c such that in some (and hence in every) local coordinate system the matrices $\| c^{ij} \|$ and $\| \sigma_{ij} \|$ are reciprocal: $c^{ij}\sigma_{jk} = \delta^i_k$.*

Proof. Let $\{f_1, f_2\}_P$ be the Poisson brackets defined by the bivector $c = \sigma^{-1}$ via (40), and let $\{f_1, f_2\}_S$ denote the Poisson brackets defined by the form σ via (32).

In local canonical coordinates $x^1, \ldots, x^n, y_1, \ldots, y_n$ the form σ looks like $\sum_i dx^i \wedge dy_i$ where $\partial_i = \partial/\partial_{x^i}$, $\partial^i = \partial/\partial_{y_i}$. The bivector $c = \sigma^{-1}$ in the same coordinate system takes the form $\sum_{i=1}^n \partial_i \wedge \partial^i$. From this we easily obtain the equalities

$$\{f_1, f_2\}_P = \partial_i f_1 \partial^i f_2 - \partial^i f_1 \partial_i f_2 = \{f_1, f_2\}_S.$$

Thus, the bivector $c = \sigma^{-1}$ defines the same Poisson bracket as the symplectic form σ. □

We return now to general Poisson manifolds. The main structure theorem about Poisson manifolds (see, e.g., [**Ki4**]) is the following

Theorem 6. *Any Poisson manifold (M, c) is foliated by the so-called* **symplectic leaves**, *i.e. can be uniquely represented as the disjoint union of submanifolds $\{M_\alpha\}_{\alpha \in A}$ such that the bivector c has a non-degenerate restriction c_α on each M_α and $(M_\alpha, c_\alpha^{-1})$ is a symplectic manifold for each $\alpha \in A$.* □

In other words, on each M_α there is the Poisson bracket $\{\cdot\, ,\ \cdot\}_\alpha$ related to a symplectic structure $\sigma_\alpha = (c_\alpha)^{-1}$ such that the value of $\{f_1, f_2\}$ at the point $m \in M_\alpha$ is equal to $\{f_1|_{M_\alpha}, f_2|_{M_\alpha}\}_\alpha(m)$.

Example 11. Define a Poisson structure on $\mathbb{R}^2$ by

$$c = f(x, y)\,\partial/\partial_x \wedge \partial/\partial_y, \qquad f \in C^\infty(\mathbb{R}^2).$$

Then the plane $\mathbb{R}^2$ splits into 2-dimensional and 0-dimensional symplectic leaves. The former are connected components of the open set $f \neq 0$, and the latter are points where $f = 0$.

3.3. Mathematical model of classical mechanics.

Symplectic geometry is the mathematical counterpart of the Hamiltonian formalism in classical mechanics. We adduce here the corresponding

physics-mathematics dictionary:

Physical notions	**Mathematical interpretations**
configuration space	smooth manifold N
position coordinates	local coordinates $q^1, \dots, q^n$ on N
phase space	cotangent bundle $M = T^*N$
impulse coordinates	coordinates $p_1, \dots, p_n$ in fibers of T^*N dual to $q^1, \dots, q^n$
state of the system	a point $m \in M$
physical observable	a smooth function f on M
the value of an observable in a given state	the value $f(m)$
kinetic energy K	a positive quadratic form K on the fibers of T^*M
potential energy V	a smooth real-valued function V on N
total energy $H = K + V$	the function $H = K + p^*(V)$ on M
equations of motion	$\dot{f} = \{f, H\}$

The remarkable fact, discovered by Hamilton, is that the final result, the equations of motion, are invariant under a very big group of symplectomorphisms of M.

Moreover, the whole theory can be formulated in the much more general situation where M is an arbitrary symplectic manifold and H is an arbitrary smooth function on M.

In particular, among these more general phase spaces one can find the classical analogue of the spin particle in quantum mechanics. It is the usual 2-dimensional sphere of integral area. We refer to [**Ki7**] for details.

3.4. Symplectic reduction.

In classical mechanics there is a useful procedure which allows us to reduce the number of degrees of freedom in the presence of symmetry.

Mathematically speaking, it is a prescription to construct a new symplectic manifold from a given symplectic manifold with a symmetry.

The initial data are the following:

G — a Lie group with the Lie algebra $\mathfrak{g}$;

(M, σ) — a symplectic G-manifold which is also a G-Poisson manifold;

$\mu : M \longrightarrow \mathfrak{g}^*$ — the moment map;

Ω_F — the coadjoint orbit in $\mathfrak{g}^*$ passing through the point $F \in \mathfrak{g}^*$;

$Stab(F)$ — the stabilizer of the point F in G, and $stab(F)$ is the Lie algebra of $Stab(F)$;

M_0 — the reduced manifold $\mu^{-1}(F)/Stab(F) \simeq \mu^{-1}(\Omega_F)/G$.

We assume that the following equivalent conditions are satisfied:

a) the group $Stab(F)$ acts freely on $\mu^{-1}(F)$.

b) the group G acts freely on $\mu^{-1}(\Omega_F)$).

Proposition 9. *The coset space M_0 is a symplectic manifold with respect to the form σ_0 defined by $p^*\sigma_0 = \sigma$, where $p : \mu^{-1}(\Omega_F) \longrightarrow M_0$ is the canonical projection.*

Proof. From our assumptions it follows that the coset space M_0 is a smooth manifold. Consider the geometric distribution $P = \ker \sigma$ on $\mu^{-1}(\Omega_F) \subset M$. The fibers of the projection p are exactly the leaves of the foliation of $\mu^{-1}(\Omega_F) \subset M$ associated with P. □

The following fact is important for application to mechanics. Assume that the function H is G-invariant. Then it defines a function H_0 on M_0.

Proposition 10. *The flow generated on M by the Hamiltonian vector field* s-grad H *preserves $\mu^{-1}(\Omega_F)$ and is projected to the flow on M_0 generated by the Hamiltonian vector field* s-grad H_0. □

This approach is based on the general principle of **symplectic reduction**. In pure mathematical terms this notion was introduced independently by Marsden-Weinstein and by Arnold in 1974, although its origin is in the classical mechanics of the 18th and 19th centuries.

The idea of symplectic reduction is very simple. A submanifold N of a symplectic manifold (M, σ) in general is not symplectic because the 2-form $\sigma' = \sigma\big|_N$ can be degenerate.

But σ' is still closed, therefore the kernel of σ' is an integrable distribution on N (it clearly satisfies the Frobenius criterion in the second formulation). If it has a locally constant rank and if the leaves of the corresponding foliation are locally closed, then we can view N as a fibration over some manifold M_0. It is clear that the form σ' can be "descended" to M_0, i.e. $\sigma\big|_N = p^*(\sigma_0)$ where σ_0 is a non-degenerate 2-form on M_0 and $p : N \longrightarrow M_0$ is the projection.

Thus, we get a smaller symplectic manifold (M_0, σ_0), which is called **reduced**.

In practical applications (M, σ) is usually interpreted as a phase space of a Hamiltonian mechanical system with the energy function $H \in C^\infty(M)$. The time development of the system is given by the Hamiltonian flow $\Phi(t) = \exp tv$ generated by the vector field $v = \text{s-grad}\, H$.

Suppose that the submanifold N can be given by a system of equations

$$F_i = 0, \quad 1 \leq i \leq k, \tag{43}$$

where the F_i are the **first integrals** of the system, i.e. smooth functions on M that Poisson commute with H.

Then the flow $\Phi(t)$ preserves the submanifold N and its foliation, hence can be descended to a flow $\Phi_0(t)$ on the reduced manifold (M_0, σ_0). Moreover, the function H is constant along the fibers and so defines a function H_0 on M_0. Finally, one can check that the restriction of the initial Hamiltonian flow to the reduced manifold is itself a Hamiltonian flow generated by the vector field $v_0 = \text{s-grad}\, H_0$.

The construction described is especially effective when the functions F_i in (43) span a Lie subalgebra $\mathfrak{g}$ in $C^\infty(M)$, viewed as a Lie algebra with respect to the Poisson bracket on M. Let us consider the F_i as coordinates on the space $\mathfrak{g}^*$, dual to $\mathfrak{g}$. Then we can interpret the collection $\{F_i\}_{1 \leq i \leq k}$ as a map μ from M to $\mathfrak{g}^*$. Following mechanical terminology, it is called a **moment map.**

This map has many nice and useful properties, which we discuss later. Here we list only some simple facts which are needed in this section.

First, this map is by definition $\mathfrak{g}$-equivariant. (Recall that $\mathfrak{g}$ acts on M via vector fields $v_i = \text{s-grad}\ F_i$, $1 \leq i \leq k$, and on $\mathfrak{g}^*$ via the coadjoint representation: $K_*(F_i)F_j = \{F_i, F_j\}$.)

Second, assume that the Lie algebra action of $\mathfrak{g}$ on M can be lifted to an action of the connected Lie group G with $\text{Lie}(G) = \mathfrak{g}$. Then the moment map μ will be G-equivariant.

Third, choose any G-orbit $\Omega \subset \mathfrak{g}^*$ and let $N = \mu^{-1}(\Omega)$. It is a G-invariant submanifold of M and one can check that the corresponding reduced symplectic manifold (M_0, σ_0) is just N/G, the set of G-orbits in N.

Lie Groups and Homogeneous Manifolds

1. Lie groups and Lie algebras

1.1. Lie groups.

A **Lie group** is a smooth manifold G endowed with a multiplication law that is a smooth map $G \times G \to G$ satisfying the usual group axioms.

Consider first the following particular case: G is a subgroup and at the same time a smooth submanifold of $GL(n, \mathbb{R})$. Such a group is usually called a **matrix Lie group**. Actually this particular case is almost a general one: every Lie group is **locally isomorphic** to a matrix Lie group. It means that there exists a diffeomorphism between some neighborhoods of the units compatible with the group law. Two locally isomorphic Lie groups have isomorphic covering groups.

We introduce convenient notation which allows us to treat general Lie groups as matrix groups. We call this **matrix notation**.

Let v be a tangent vector to G at a point $x \in G$. The left and right shifts by an element $g \in G$ are smooth maps of G to itself. The corresponding derivative maps send the space T_xG to $T_{gx}G$ and $T_{xg}G$. We denote the image of v under these maps by $g \cdot v$ and $v \cdot g$, respectively.

Note that for a matrix Lie group $G \subset GL(n, \mathbb{R})$ the tangent space T_xG can be identified with a subspace in $\mathrm{Mat}_n(\mathbb{R})$ and the expressions $g \cdot v$ and $v \cdot g$ can be understood literally as products of matrices.

We shall also use matrix notation for covectors: for $f \in T_x^*G$ we denote by $g \cdot f$ the covector in $T^*_{x \cdot g^{-1}}G$ defined by $\langle g \cdot f\,,\, v\rangle = \langle f\,,\, v \cdot g\rangle$. In the same way we put $\langle f \cdot g\,,\, v\rangle = \langle f\,,\, g \cdot v\rangle$ for $f \in T_x^*G,\ v \in T^*_{g^{-1}x}G$.

For a matrix Lie group $G \subset GL(n,\, \mathbb{R})$ the cotangent space T_x^*G is identified with a factor space of $\mathrm{Mat}_n(\mathbb{R})$ modulo $T_xG^{\perp}$ using the pairing

$$\langle A,\, B\rangle = \mathrm{tr}\,(AB).$$

Here again the expressions $g \cdot f$ and $f \cdot g$ can be understood as products of a matrix g and a class of matrices $f \in \mathrm{Mat}_n(\mathbb{R})/T_xG^{\perp}$. Indeed,

$$\langle g \cdot f,\, v\rangle = \mathrm{tr}\,(gf)v = \mathrm{tr}\, f(vg) = \langle f,\, v \cdot g\rangle,$$
$$\langle f \cdot g,\, v\rangle = \mathrm{tr}\,(fg)v = \mathrm{tr}\, f(gv) = \langle f,\, g \cdot v\rangle.$$

1.2. Lie algebras.

A Lie group is a rather complicated non-linear object. Fortunately, it can be almost uniquely determined by a linear object, the so-called Lie algebra.

By definition, a **Lie algebra** is a vector space over some field K (in our book we shall use only real and complex fields) endowed with a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

This map is called the **commutator** and is usually denoted by brackets $[X,\, Y]$. By definition, it satisfies the following conditions:

Antisymmetry: $[X,\, Y] = -\,[Y,\, X]$ (or, equivalently, $[X,\, X] = 0$);

Jacobi identity:

$$\circlearrowleft [X,\, [Y,\, Z]] := [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \tag{1}$$

Here and below the sign $\circlearrowleft$ is used for the summation over cyclic permutations of arguments.

The definition of a Lie algebra, especially the Jacobi identity, at first sight looks rather cumbersome and far-fetched. We give two important interpretations of this identity here that are easier to memorize:

a) The map $\mathrm{ad}\, X:\ Y \mapsto [X,\, Y]$ is a derivation of $\mathfrak{g}$, i.e. satisfies the Leibniz rule:

$$\mathrm{ad}\, X\,([Y,\, Z]) = [\mathrm{ad}\, X\,(Y),\, Z] + [Y,\, \mathrm{ad}\, X\,(Z)]. \tag{2}$$

b) The map $\mathrm{ad}:\ X \mapsto \mathrm{ad}\, X$ is a Lie algebra homomorphism of $\mathfrak{g}$ to $\mathrm{End}\,\mathfrak{g}$, i.e.

$$\mathrm{ad}\,[X,\, Y] = \mathrm{ad}\, X \circ \mathrm{ad}\, Y - \mathrm{ad}\, Y \circ \mathrm{ad}\, X. \tag{3}$$

In fact, the definition of a Lie algebra is well justified. The main reason will be given in the next section, but meanwhile we present an argument which shows that Lie algebras are objects at least as natural as associative commutative algebras.

Proposition 1. *Let $\mathcal{A}$ be a real algebra with the multiplication law denoted by $*$.*

a) *Suppose that in $\mathcal{A}$ some non-trivial bilinear identity holds:*

$$\alpha \cdot (a * b) + \beta \cdot (b * a) \equiv 0$$

where α, β are some fixed real numbers and a, b denote arbitrary elements of $\mathcal{A}$. Then $\mathcal{A}$ is either commutative or anticommutative.

b) *Assume further that $\mathcal{A}$ admits some non-trivial trilinear identity:*

$$\lambda \cdot a * (b * c) + \mu \cdot b * (c * a) + \nu \cdot c * (a * b) \equiv 0$$

where λ, μ, ν are fixed real numbers and a, b, c denote arbitrary elements of $\mathcal{A}$.

Then, in the commutative case, $\mathcal{A}$ is either an associative commutative algebra or a non-associative commutative algebra where the identity $x^3 \equiv 0$ is satisfied.

In the anticommutative case either $\mathcal{A}$ is a Lie algebra, or an algebra where the product of any four elements is zero.

Proof. a) By interchanging a and b in the first relation we get the system of linear equations

$$\begin{cases} \alpha \cdot a * b + \beta \cdot b * a \equiv 0, \\ \beta \cdot a * b + \alpha \cdot b * a \equiv 0. \end{cases}$$

If the multiplication is identically zero, we have nothing to prove. Otherwise, the system has non-trivial solutions and its determinant must be zero. We get $\alpha = \pm\beta$, hence our multiplication is either commutative or anticommutative.

b) Let us apply a similar argument to the trilinear relation. Namely, consider the system

$$\begin{cases} \lambda \cdot a * (b * c) + \mu \cdot b * (c * a) + \nu \cdot c * (a * b) = 0, \\ \nu \cdot a * (b * c) + \lambda \cdot b * (c * a) + \mu \cdot c * (a * b) = 0, \\ \mu \cdot a * (b * c) + \nu \cdot b * (c * a) + \lambda \cdot c * (a * b) = 0. \end{cases}$$

The determinant of this system is

$$\lambda^3 + \mu^3 + \nu^3 - 3\lambda\mu\nu = (\lambda + \mu + \nu)(\lambda + \epsilon\mu + \epsilon^2\nu)(\lambda + \epsilon^2\mu + \epsilon\nu)$$

where $\epsilon = e^{\frac{2\pi i}{3}}$ is a cubic root of unity.

If $a * (b * c)$ is not identically zero, this determinant must vanish. Note that the last two factors of the determinant are complex conjugate. So we have the following alternative:[1]

[1]This is related to the following representation-theoretic fact: the space $\mathbb{R}^3$ under the action of the permutation group S_3 splits into two irreducible subspaces: the one-dimensional space $L_1 = \{(\lambda, \lambda, \lambda)\}$ and the two-dimensional space $L_2 = L_1^{\perp} = \{(\lambda, \mu, \nu) \mid \lambda + \mu + \nu = 0\}$.

Either $(\lambda + \epsilon\mu + \epsilon^2\nu) = (\lambda + \epsilon^2\mu + \epsilon\nu) = 0$, hence $\lambda = \mu = \nu$ and we get the identity

$$a * (b * c) + b * (c * a) + c * (a * b) \equiv 0, \tag{4}$$

or the trilinear relation holds for any λ, μ, ν with $\lambda + \mu + \nu = 0$ and we get

$$a * (b * c) = b * (c * a) = c * (a * b). \tag{5}$$

In the commutative case (4) implies $x^3 = 0$. Conversely, if the identity $x^3 \equiv 0$ holds in a commutative algebra $\mathcal{A}$, then, putting $x = \alpha a + \beta b + \gamma c$ and comparing coefficients for $\alpha\beta\gamma$, we get the relation (4).

The identity (5) in the commutative case gives the associativity law.

In the anticommutative case (4) is the Jacobi identity, while (5) means that the triple product $a * (b * c)$ is totally antisymmetric.

Let L_a (resp. R_a) denote the operator of left (resp. right) multiplication by a in $\mathcal{A}$. The antisymmetry of the product $a*b$ and the total antisymmetry of the triple product $a * (b * c)$ implies the relations:

$$R_a = -L_a; \qquad L_aL_b = -L_bL_a; \qquad L_{a*b} = L_bL_a.$$

From these relations we deduce

$$L_aL_bL_c = -L_{a*b}L_c = L_cL_{a*b} = L_cL_bL_a = -L_aL_bL_c,$$

hence $L_aL_bL_c \equiv 0$ and the product of every four elements is zero. □

An important class of Lie algebras is formed by **matrix Lie algebras** that are subspaces of $\mathrm{Mat}_n(K)$ closed with respect to the ordinary matrix commutator

$$[X, Y] = XY - YX.$$

Both conditions above can be easily checked.

Actually, this class is universal because of

Ado's Theorem. *Any Lie algebra is isomorphic to a matrix Lie algebra.*□

The example below historically was the first appearance of a Lie algebra in mathematics.

We start with a definition. An algebra A is called a **division algebra** if any non-zero element of A is invertible. Associative non-commutative division algebras are also called **skew fields**.

Frobenius' Theorem. *There are only three (up to isomorphism) associative division algebras over $\mathbb{R}$: the real field $\mathbb{R}$, the complex field $\mathbb{C}$, and a skewfield $\mathbb{H}$ of* **quaternions**. □

The latter, very remarkable, algebra was discovered by W. Hamilton in 1843 after many years of unsuccessful attempts to generalize complex numbers and define an associative multiplication in $\mathbb{R}^3$. The crucial idea was to switch from $\mathbb{R}^3$ to $\mathbb{R}^4$ (and also to sacrifice the commutativity law). We describe this algebra below.

Hamilton represented a quaternion in the form $\mathbf{q} = x_0 + \mathbf{x}$ where x_0 is an ordinary real number and $\mathbf{x} = (x_1, x_2, x_3)$ is a vector in $\mathbb{R}^3$; he called these constituents the **scalar** and **vector** parts of $\mathbf{q}$. The product of two scalars and the multiplication of a vector by a scalar are as usual. So, one needs only to define a product of two vectors. This product $\mathbf{x} \cdot \mathbf{y}$ splits into a scalar and a vector part, which were called by Hamilton the **scalar product** $(\mathbf{x}, \mathbf{y})$ and the **vector product** $\mathbf{x} \times \mathbf{y}$ (these terms and notation are still presently used).

The explicit formulae are:

$$\mathbf{x} \cdot \mathbf{y} = (\mathbf{x}, \mathbf{y}) + \mathbf{x} \times \mathbf{y},$$

$$(\mathbf{x}, \mathbf{y}) = x_1 y_1 + x_2 y_2 + x_3 y_3, \qquad \mathbf{x} \times \mathbf{y} = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are standard basic vectors in $\mathbb{R}^3$.

The algebra $\mathbb{H}$ can be conveniently realized as a subalgebra of $\mathrm{Mat}_2(\mathbb{C})$ via

$$\mathbf{q} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \quad \longleftrightarrow \quad \begin{pmatrix} x_0 + ix_3 & x_1 + ix_2 \\ -x_1 + ix_2 & x_0 - ix_3 \end{pmatrix}.$$

It can also be defined as an associative algebra spanned by one real unit 1 and three imaginary units $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfying the relations[2]

$$\text{(H)} \qquad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i} \cdot \mathbf{j} \cdot \mathbf{k} = -1.$$

Example 1. The 3-dimensional Euclidean vector space $\mathbb{R}^3$ endowed with vector multiplication is a real Lie algebra. ◇

Exercise 1. a) Check that the operation of vector multiplication satisfies all the axioms of a Lie algebra.

[2]The legend is that these very relations were carved by Hamilton on the railings of a bridge he usually crossed during his mathematical walks.

b)* Show that the Jacobi identity implies the following well-known fact from Euclidean geometry: for any triangle ABC the three altitudes are concurrent.[3]

Hint. Consider points and lines in the Euclidean plane $\mathbb{R}^2$ in terms of homogeneous coordinates on $\mathbb{P}^2(\mathbb{R})$: a point $(x, y) \in \mathbb{R}^2$ corresponds to $(x : y : 1) \in \mathbb{P}^2(\mathbb{R})$, and the line $ax+by+c=0$ corresponds to $(a:b:c) \in \mathbb{P}^2(\mathbb{R})$.

Note that the point $l_\infty = (0:0:1)$ does not correspond to any line in $\mathbb{R}^2$. It is related to the "infinite line" $\mathbb{P}^2(\mathbb{R})\backslash\mathbb{R}^2$. The solution to b) follows from the statements:

(i) A point $x = (x_0 : x_1 : x_2)$ belongs to a line $a = (a_0 : a_1 : a_2)$ iff

$$(a,\, x) := \sum_{i=1}^{3} a_i x_i = 0.$$

(ii) The line a passing through points x and y is given by $a = x \times y$.

(iii) The intersection point x of two lines a and b is given by $x = a \times b$.

(iv) The line p passing through a point x and perpendicular to a line a is given by $p = x \times \tilde{a}$ where $\tilde{a} = a - l_\infty \cdot \frac{(a,\, l_\infty)}{(l_\infty,\, l_\infty)}$.

(v) Three lines a, b, c are concurrent iff their **mixed product**

$$(a,\, b,\, c) := (a \times b,\, c) = \det \begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix}$$

vanishes (in other words, the vectors a, b, c are linearly dependent).

(vi) $\circlearrowleft_{x,y,z} (x,y,a)(z,b,c) = (x,y,z)(a,b,c)$ where the sign $\circlearrowleft_{x,y,z}$ denotes the sum over cyclic permutations of three vectors x, y, z. ♣

1.3. Five definitions of the functor Lie: $G \rightsquigarrow \mathfrak{g}$.

To any Lie group G one can canonically define the Lie algebra $\mathfrak{g} = \mathrm{Lie}\,(G)$. In more rigorous terms, there is a functor from the category of Lie groups (objects are Lie groups, morphisms are smooth group homomorphisms) to the category of Lie algebras (objects are Lie algebras, morphisms are Lie algebra homomorphisms).

This functor can be defined in many equivalent ways. We give five different constructions here, each of which has its own advantage.

[3]This fact is mentioned in the recent paper [**Ar1**] which is of independent interest. It is worthwhile to mention that the proof indicated becomes even simpler in the case of spherical geometry: the line through the point x perpendicular to the line a is given by $x \times a$ (cf. (iv) below).

All definitions will be illustrated on the example $G = \mathrm{Aff}(1, \mathbb{R})$, the Lie group of all affine transformations of the real line $\mathbb{R}^1$ with parameter t.

To define the functor Lie we have to associate to any Lie group G some Lie algebra $\mathfrak{g} = \mathrm{Lie}\,(G)$ and to any smooth group homomorphism $\Phi : G \to H$ some Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$.

As a vector space, we define $\mathfrak{g}$ as the tangent space T_eG to the group G at the unit element e. The map φ is just the derivative of Φ at e. It remains to endow $\mathfrak{g}$ with the Lie algebra structure, i.e. define the commutator. We do it below in five different ways.

1. Let us write the multiplication law on G in the coordinate form. It is given by n functions of $2n$ variables expressing the coordinates $\{z^k\}$ of the product in terms of coordinates $\{x^i\}$, $\{y^j\}$ of the factors: $z^k = f^k(x^1, \dots, x^n;\, y^1, \dots, y^n)$.

Assume that our local coordinate system is centered at the unit element. Then we have $f^k(x^1, \dots, x^n; 0, \dots, 0) = x^k$, $f^k(0, \dots, 0; y^1, \dots, y^n) = y^k$. We conclude that the Taylor decomposition of f^k at the origin has the form[4]

$$f^k(x^1, \dots, x^n;\, y^1, \dots, y^n) = x^k + y^k + b^k_{ij}x^iy^j + \text{higher order terms.} \tag{6}$$

Now we introduce the quantities

$$c^k_{ij} := b^k_{ij} - b^k_{ji} \tag{7}$$

and consider them as structure constants of the desired Lie algebra $\mathfrak{g}$.

This means that the commutators of basic vectors $X_1, \dots, X_n$ in $\mathfrak{g}$ are given by the formula

$$[X_i,\, X_j] = c^k_{ij}X_k. \tag{8}$$

Let us see how this construction works for the example $G = \mathrm{Aff}(1, \mathbb{R})$. The general affine transformation of $\mathbb{R}^1$ has the form

$$\phi_{a,b} :\ t \mapsto at + b \quad \text{where } a \in \mathbb{R}\backslash\{0\},\ b \in \mathbb{R}.$$

Hence, the whole group can be covered by one (non-connected) chart with coordinates (a, b). To put the origin at the unit element, we choose another coordinate system: $x^1 = a - 1$, $x^2 = b$, and define the transformation ψ_{x^1,x^2} as $t \mapsto (x^1 + 1)t + x^2$.

[4] Recall that we use the Einstein notation (see Remark 1 in Appendix II): if an index appears twice, once as a lower index and once as an upper one, then the summation over this index is understood.

The composition of transformations ψ_{x^1,x^2} and ψ_{y^1,y^2} has the form ψ_{z^1,z^2} with

$$z^1 = x^1 + y^1 + x^1y^1, \qquad z^2 = x^2 + y^2 + x^1y^2.$$

We see that the only non-zero coefficients b_{ij}^k are $b_{11}^1 = b_{12}^2 = 1$. Hence, the only non-zero structure constants are $c_{12}^2 = -c_{21}^2 = 1$.

The basic commutation relation is $[X_1, X_2] = X_2$.

2. Let X, Y be any two tangent vectors to G at e. Define their commutator $[X,Y]$ as follows. Let $\phi(\tau)$ and $\psi(\tau)$ be any smooth curves in G with the properties

$$\phi(0) = \psi(0) = e; \qquad \dot{\phi}(0) = X, \qquad \dot{\psi}(0) = Y.$$

Then the curve

$$\xi(\tau) = \phi(\sqrt{\tau})\psi(\sqrt{\tau})\phi(\sqrt{\tau})^{-1}\psi(\sqrt{\tau})^{-1} \tag{9}$$

is 1-smooth for $\tau \geq 0$ and we put $[X,Y] = \dot{\xi}(0)$.

In the example let $X_1 = \partial_1 := \frac{\partial}{\partial x^1}$, $X_2 = \partial_2 := \frac{\partial}{\partial x^2}$. As representatives of these vectors we can choose the curves

$$\phi(\tau) = \psi_{\tau,0} \colon t \mapsto (1+\tau)t, \qquad \psi(\tau) = \psi_{0,\tau} \colon t \mapsto t + \tau.$$

Then the transformation (5) is $t \mapsto (1+\sqrt{\tau})\left(\frac{t-\sqrt{\tau}}{1+\sqrt{\tau}} + \sqrt{\tau}\right) = t + \tau$, i.e. coincides with $\psi(\tau)$. We come again to the commutation relation $[X_1, X_2] = X_2$.

3. For $g \in G$ consider the map $A(g) : G \to G : h \mapsto g\,h\,g^{-1}$. It is a so-called **inner automorphism** of G. In particular, it preserves the multiplication law and fixes the neutral element e.

Put $\operatorname{Ad} g := A(g)_*(e)$, the derivative of $A(g)(\cdot)$ at the point e. This is a linear transformation of the space $\mathfrak{g} = T_e(G)$. Moreover, the chain rule for derivatives implies that $\operatorname{Ad} g_1 \operatorname{Ad} g_2 = \operatorname{Ad} g_1 g_2$. Thus, the map $g \mapsto \operatorname{Ad} g$ is a linear representation of G in the space $\mathfrak{g}$, called the **adjoint representation**.

Now, Ad is a smooth map from G to the group $\operatorname{Aut}\mathfrak{g}$ of all automorphisms of the vector space $\mathfrak{g}$. Let us denote by ad the derivative of this map at e. It is a map from $\mathfrak{g} = T_eG$ to $\operatorname{End}\mathfrak{g} = T_{\mathrm{Id}}\operatorname{Aut}\mathfrak{g}$. So, for any $X \in \mathfrak{g}$ the quantity $\operatorname{ad} X := \operatorname{Ad}_*(e)(X)$ is a linear operator in $\mathfrak{g}$.

Finally, we put $[X,Y] = (\operatorname{ad} X)(Y)$.

In the example let $g = \psi_{x^1,x^2}$, $\quad h = \psi_{y^1,y^2}$. Then the transformation $A(g)$ sends h to ghg^{-1}: $t \mapsto (1+y^1)t + (1+x^1)y^2 - x^2y^1$ (check it!).

Hence, in coordinates (y^1, y^2) on G the map $A(g)$ is a linear transformation:

$$A(g) = \begin{pmatrix} 1 & 0 \\ -x^2 & 1+x^1 \end{pmatrix}.$$

Therefore, its derivative $\operatorname{Ad} g$ is given by the same formula, if we identify the neighborhood of the unity in G with a neighborhood of the origin in $\mathfrak{g}$ using the coordinate system (y^1, y^2).

Put as above $X_i = \partial_i = \frac{\partial}{\partial x^i}$, $i = 1, 2$. Then we get:

$$\operatorname{ad} X_1 = \frac{\partial \operatorname{Ad} g}{\partial x^1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \operatorname{ad} X_2 = \frac{\partial \operatorname{Ad} g}{\partial x^2} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

We come once again to the commutation relation $[X_1, X_2] = X_2$.

4. To any vector $X \in \mathfrak{g}$ there corresponds the unique left-invariant vector field $\widetilde{X} \in \mathit{Vect}\, G$, which takes the value X at e. Since the Lie bracket is a natural operation, it commutes with all diffeomorphisms and, in particular, with group translations. Therefore, the Lie bracket of two left-invariant vector fields is also left-invariant. So, $[\widetilde{X}, \widetilde{Y}]$ has the form $\widetilde{Z}$ for some $Z \in \mathfrak{g}$. We then define $[X, Y]$ as Z.

Warning. If we use the right-invariant vector fields $\widehat{X} \in \mathit{Vect}\, G$ that take the value X at e instead of left-invariant fields $\widetilde{X}$, we get a different definition of the commutator, the negative of the previous one. The reason is that the left-invariant vector fields are generators of right shifts on G and the right-invariant vector fields generate left shifts (see below). So, the latter give a representation of $\mathfrak{g}$ while the former give an antirepresentation.

To get the same Lie algebra structure we have to associate to $X \in \mathfrak{g}$ the right-invariant field $-\widehat{X}$. ♠

On the group $G = \operatorname{Aff}(1, \mathbb{R})$ with coordinates (a, b) the general vector field has the form $v = A(a, b)\partial_a + B(a, b)\partial_b$. The left shift by the element $\phi_{\alpha,\beta}$ sends the point (a, b) to the point $(a', b') = (\alpha a, \alpha b + \beta)$. We have

$$da' = \alpha da, \quad db' = \alpha db, \qquad \text{and} \qquad \frac{\partial}{\partial a'} = \alpha^{-1}\frac{\partial}{\partial a}, \quad \frac{\partial}{\partial b'} = \alpha^{-1}\frac{\partial}{\partial b}.$$

Hence, the above field v goes to

$$v' = A(a', b')\partial_{a'} + B(a', b')\partial_{b'} = \frac{A(\alpha a, \alpha b + \beta)}{\alpha}\partial_a + \frac{B(\alpha a, \alpha b + \beta)}{\alpha}\partial_b.$$

Therefore, the field v is left-invariant iff

$$A(\alpha a, \alpha b + \beta) = \alpha A(a, b), \quad B(\alpha a, \alpha b + \beta) = \alpha B(a, b).$$

The general solution to these equations is $A(a, b) = c_1 a$, $B(a, b) = c_2 a$. We get

$$\widetilde{X}_1 = a\partial_a, \qquad \widetilde{X}_2 = a\partial_b.$$

Note that the same result can be obtain using the following principle:

a left-invariant vector field is a generator of a right shift :

$$\exp \widetilde{X}\,(g) = g \cdot \exp X.$$

We leave it to the reader to derive the following formula for right-invariant fields on $\mathrm{Aff}(1, \mathbb{R})$ (which are generators of left shifts):

$$\widehat{X}_1 = a\partial_a + b\partial_b, \qquad \widehat{X}_2 = \partial_b.$$

It is clear that $[\widetilde{X}_1, \widetilde{X}_2] = \widetilde{X}_2$, while $[\widehat{X}_1, \widehat{X}_2] = -\widehat{X}_2$.

5. Consider any matrix (or operator) realization π of G by operators on $\mathbb{R}^n$. Then $\pi(G)$ is a submanifold in $\mathrm{Mat}_n(\mathbb{R})$. Let $\mathfrak{g}$ be the tangent vector space[5] to $\pi(G)$ at the point 1_n. Define the commutator in $\mathfrak{g}$ by the formula $[X, Y] = XY - YX$.

In our example we can use the following matrix realization of the group $G = \mathrm{Aff}(1, \mathbb{R})$:

$$\{x \mapsto ax + b\} \quad \longleftrightarrow \quad \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

Indeed, the equality

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax + b \\ 1 \end{pmatrix}$$

shows that the standard linear action of the matrix $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ on column vectors $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ defines a standard affine action on the line $y = 1$.

The tangent space $\mathfrak{g} = T_e G$ is the space of matrices of the form

$$X = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{R},$$

with the natural basis

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

[5] Do not mix the linear subspace $\mathfrak{g} \subset \mathrm{Mat}_n(\mathbb{R})$ with the geometric tangent space that is the affine manifold $1_n + \mathfrak{g} \subset \mathrm{Mat}_n(\mathbb{R})$.

Here we get once more the commutation relation $[X_1, X_2] = X_2$.

Let us discuss the five definitions above. First, we mention their positive features.

The first and last definitions are the most convenient for practical computations (see examples below).

The second is more geometric and shows the relation between the Lie group commutator and the Lie algebra commutator: the latter is a limit of the former.

The third definition is the most conceptual (it contains no computations at all) and therefore is the most convenient in theoretical questions.

The fourth definition introduces a useful notion of left- (or right-) invariant vector fields on G. They are particular cases of left- (or right-) invariant differential operators on G which play an important role in harmonic analysis on Lie groups.

On the other hand, the first definition has a serious disadvantage: it depends on the choice of local coordinates. Consider this dependence in more detail.

Note that under a linear transformation of local coordinates, the quantity b_{ij}^k behaves like a tensor of type $(2,1)$. If, however, we make a non-linear transformation $x^k \mapsto x^k + a_{ij}^k x^i x^j + \cdots$, then b_{ij}^k changes in a more complicated affine way, namely

$$b_{ij}^k \mapsto b_{ij}^k - 2a_{ij}^k x^i y^j.$$

The remarkable fact is that the quantities c_{ij}^k still behave like coordinates of a tensor. (In particular, under the above transformation they do not change at all.)

Hence, a bilinear operation $[x,y]^k = c_{ij}^k x^i y^j$ is correctly defined on the tangent space T_eG.

There is another argument in favor of c_{ij}^k comparing with b_{ij}^k. Let $\{x^1, \ldots, x^n\}$ be a local coordinate system. Call such a system **symmetric** if $x^k(g^{-1}) = -x^k(g)$. The following lemma shows the existence of symmetric coordinate systems and describes their useful properties.

Lemma 1. *Let $\{x^1, \ldots, x^n\}$ be any local coordinate system centered at e. Introduce the functions $\widetilde{x}^k(g) := \frac{x^k(g) - x^k(g^{-1})}{2}$. Then*

a) *The functions $\{\widetilde{x}^1, \ldots, \widetilde{x}^n\}$ form a symmetric coordinate system.*

b) *The Jacobi matrix $\|\frac{\partial \widetilde{x}^k}{\partial x^j}\|$ is equal to 1_n at the origin.*

c) *In any symmetric local coordinate system the quantities b_{ij}^k are anti-symmetric with respect to i,j and therefore coincide with $\frac{1}{2}c_{ij}^k$.*

d) *The group law in symmetric local coordinates looks like*

$$(10)\quad f(x;y) = x+y+\frac{1}{2}[x,\, y]+T_1(x,\, x,\, y)+T_2(x,\, y,\, y)+ \textit{terms of order} \geq 4$$

where $[x,\, y]^k = c^k_{ij}x^i y^j$ *and* T_i, $i = 1, 2$, *are trilinear vector-valued forms satisfying the conditions:*

(i) $T_1(x,\, y,\, z) = T_1(y,\, x,\, z)$, $\qquad T_2(x,\, y,\, z) = T_2(x,\, z,\, y)$,

(ii) $T_1(x,\, x,\, x) = T_2(x,\, x,\, x)$.

Proof. a) is clear, provided we know that $\{\widetilde{x}^k(g)\}$ is a coordinate system.

b) In any coordinate system centered at e we have $x^k(g^{-1}) = -x^k(g)+$ higher order terms. So, $\|\frac{\partial \widetilde{x}^k}{\partial x^j}\|(e) = 1$. It also proves that $\{\widetilde{x}^k(g)\}$ is indeed a local coordinate system.

c) Note that a symmetric local coordinate system is necessarily centered at e, so equation (6) holds. Substituting $y = -x$ and using the property $f(x;\, -x) = 0$ we conclude that $b^k_{ij}x^i x^j \equiv 0$. Hence, the coefficients b^k_{ij} are antisymmetric in i, j.

d) Consider the term of third order in the Taylor decomposition of $f(x;\, y)$. It can be written in the form $\sum_{i=0}^{3} T_i$ where $\deg_y T_i = i$.

Since $f(x;\, 0) = x$, $f(0;\, y) = y$, we have $T_0 = T_3 = 0$. It follows that (10) holds together with condition (i). Again using the property $f(x;\, -x) = 0$, we obtain $T_1(x,\, x,\, -x) + T_2(x,\, -x,\, -x) = 0$, which implies (ii). $\square$

Now we are ready to check the Jacobi identity for the first definition of Lie (G).[6] The ultimate reason for the Jacobi identity is the associativity of the group law. In terms of coordinates the associativity property has the form:

$$f(f(x;\, y);\, z) = f(x;\, f(y;\, z)).$$

We assume that our coordinate system is symmetric and compare the trilinear terms in the Taylor decompositions of the left- and right-hand sides. We get

$$\frac{1}{4}[[x,\, y],\, z] + 2T_1(x,\, y,\, z) = \frac{1}{4}[x,\, [y,\, z]] + 2T_2(x,\, y,\, z).$$

Put commutators on one side, trilinear forms T_1, T_2 on the other, and take the sum over cyclic permutations of x, y, z (denoted by $\circlearrowleft$). We obtain

$$\circlearrowleft [[x,\, y],\, z] = 4 \circlearrowleft T_1(x,\, y,\, z) - 4 \circlearrowleft T_2(x,\, y,\, z).$$

Using equations (i) and (ii) above we conclude that the right-hand side is actually zero and we get the Jacobi identity. $\square$

[6] Note that this check is very easy for the last definition and more involved for the others.

Theorem 1. *All five definitions of the commutator in* $\mathfrak{g} = \mathrm{Lie}\,(G) = T_eG$ *given above lead to isomorphic Lie algebras.*

Proof. The simplest way to establish the isomorphism of all five variants of $\mathfrak{g}$ is to compare definitions 2–5 with definition 1 using the appropriate local coordinate system. Let us do this.

2. $\Longleftrightarrow$ 1. Choose two vectors $X,\, Y \in \mathfrak{g}$ and consider two curves in G, which in some symmetric local coordinate system are given by $\phi(\tau) = \tau X$, $\psi(\tau) = \tau Y$. Then

$$\begin{aligned}\xi(\tau) &:= \phi(\sqrt{\tau})\psi(\sqrt{\tau})\phi(\sqrt{\tau})^{-1}\psi(\sqrt{\tau})^{-1} = \phi(\sqrt{\tau})\psi(\sqrt{\tau})\phi(-\sqrt{\tau})\psi(-\sqrt{\tau})\\ &= \sqrt{\tau}X + \sqrt{\tau}Y + \frac{1}{2}\tau[X,\, Y] - \sqrt{\tau}X - \frac{1}{2}\tau[Y,\, X] - \sqrt{\tau}Y + o(\tau)\\ &= \tau[X,\, Y] + o(\tau).\end{aligned}$$

3. $\Longleftrightarrow$ 1. Suppose $g \in G$ has coordinates ϵX and $h \in G$ has coordinates δY in some symmetric coordinate system. Then ghg^{-1} has coordinate $\delta Y + \frac{1}{2}\epsilon\delta[X,\, Y] +$ higher order terms. To compute $\mathrm{Ad}\; g := A(g)_*(e)$ we have to take the term that is linear in δ. So, we get

$$\mathrm{Ad}\, g\, Y = \epsilon[X,\, Y] + \text{higher order terms}.$$

Further, to compute $\mathrm{ad}\, Y := \mathrm{Ad}_*(e)$ we must take the term which is linear in ϵ. So, $\mathrm{ad}\, X\, Y = [X,\, Y]$ and we are done.

4. $\Longleftrightarrow$ 1. Let use the fact quoted above: all left-invariant vector fields are generators of right shifts and vice versa. The right shift on an element g with coordinate δY is

$$X \mapsto X + \delta Y + \frac{1}{2}\delta[X,\, Y] + \text{higher order terms}.$$

Therefore, the corresponding vector field is given by

$$v_Y^k(x) = \left(y^k + \frac{1}{2}c_{ij}^k x^i y^j\right)\partial_k + \text{higher order terms}.$$

From this we get $[v_Y,\, v_Z](0) = [Y,\, Z]$. Hence, $[v_Y,\, v_Z] = v_{[Y,\, Z]}$.

5. $\Longleftrightarrow$ 1. Let $G \subset GL(n,\, \mathbb{R})$ be a matrix group, and let

$$\mathfrak{g} := T_1G \subset T_1GL(n,\, \mathbb{R}) = \mathrm{Mat}_n(\mathbb{R})$$

be its tangent space.

We shall use the functions $\exp X$ and $\log g$, which are defined for the matrix arguments X and g by the series:

$$\exp X = \sum_{k\geq 0}\frac{X^k}{k!}, \qquad \log g := -\sum_{k\geq 0}\frac{(1-g)^k}{k} \quad (\text{defined for } \|g-1\| < 1).$$

We claim that these functions map $\mathfrak{g}$ to G and some open part of G, defined by the inequality $\|g-1\|<\epsilon$, to $\mathfrak{g}$, respectively.

Indeed, if $X\in\mathfrak{g}$, then there exists a smooth curve $\{g(t),\ t\in\mathbb{R}\}\subset G$ such that $g(t)=1+tX+o(t)$. Consider the sequence $g_n=g(\frac{1}{n})=1+\frac{X}{n}+o\left(\frac{1}{n}\right)$. The element g_n^n belongs to G and we have

$$\lim_{n\to\infty} g_n^n = \lim_{n\to\infty}\exp\left(n\log g_n\right) = \exp\left(\lim_{n\to\infty} n\left(\frac{X}{n}+o\left(\frac{1}{n}\right)\right)\right) = \exp X.$$

Since G is closed, $\exp X\in G$.

Conversely, let $U_n\subset G$ be a neighborhood of the unit defined by $\|g-1\|<\frac{1}{n}$. We temporarily assume that for any $n\in\mathbb{N}$ there exists $g_n\in U_n$ such that $\log g_n\notin\mathfrak{g}$ and come to a contradiction.

Let us split $\mathrm{Mat}_n(\mathbb{R})$ in the form $\mathrm{Mat}_n(\mathbb{R})=\mathfrak{g}\oplus V$ where V is any subspace complementary to $\mathfrak{g}$. The map $\varphi(X\oplus Y)=\exp X\cdot\exp Y,\ X\in\mathfrak{g},\ Y\in V$, has the Jacobian 1 at $(0\oplus 0)$. Hence, it establishes a diffeomorphism between a neighborhood of $0\in\mathrm{Mat}_n(\mathbb{R})$ and a neighborhood of $1\in GL(n,\,\mathbb{R})$.

We conclude that any matrix $g\in GL(n,\,\mathbb{R})$ that is sufficiently close to 1 can be uniquely written in the form $g=\exp X\cdot\exp Y,\ X\in\mathfrak{g},\ Y\in V$.

In particular, for the sequence g_n above we eventually get

$$g_n=\exp X_n\cdot\exp Y_n,\qquad X_n\in\mathfrak{g},\ Y_n\in V.$$

Note that $g_n':=\exp Y_n=\exp(-X_n)g_n\in G$, $Y_n\neq 0$ and $\|Y_n\|\to 0$ when $n\to\infty$. We can choose a subsequence $\{Y_{n_k}\}$ and integers m_k so that $m_k\cdot Y_{n_k}\to Y\in V\backslash 0$ for $k\to\infty$. Then $Y=\lim m_k\log g_{n_k}'\in\mathfrak{g}$ and at the same time $Y\in V\backslash 0$, a contradiction.

Hence, there exists a neighborhood of unit U_n for which $\log U_n\subset\mathfrak{g}$. This allows us to introduce the "logarithmic" coordinate system $x(g)=\log g\in\mathfrak{g}$ in U_n. Traditionally, this system is called **exponential**. It is evidently symmetric and in this system we have

$$\begin{aligned} f(x;\,y) &= \log\left(e^x\cdot e^y\right)\\ &= \log\left(1+x+\frac{1}{2}x^2+y+xy+\frac{1}{2}y^2+\text{higher order terms}\right)\\ &= x+y+\frac{1}{2}(xy-yx)+\text{higher order terms}. \end{aligned}$$

Hence, $[x,\,y]=xy-yx$. □

Consider again the adjoint representation Ad introduced in the third definition of $\mathfrak{g}$. When G is a matrix group, the adjoint representation is

simply the matrix conjugation:

$$\mathrm{Ad}\,(g)\,X = g \cdot X \cdot g^{-1}, \qquad X \in \mathfrak{g}, \quad g \in G. \tag{11}$$

The same formula holds in the general case if we accept the matrix notation introduced in Section 1.1.

The proof of the Jacobi identity, or, more precisely, of its interpretations (2) and (3) above, can be derived from the following very useful relation between Lie groups and Lie algebras.

Theorem 2. *There exists a unique* **exponential map** $\exp : \mathfrak{g} \to G$ *that has the following properties:*

a) *For any $X \in \mathfrak{g}$ the curve $g_X(t) = \exp tX$ is a 1-parameter subgroup in G satisfying*

$$g_X(t)g_X(s) = g_X(t+s), \qquad g_X(0) = e, \qquad \dot{g}_X(0) = X.$$

b) *For any Lie group homomorphism $\Phi\colon\ G \to H$ the following diagram is commutative:*

$$\begin{tikzcd} G \arrow[r, "\Phi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\Phi_*(e)"'] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

c) *If G is connected and simply connected, then for any Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ there exists a unique Lie group homomorphism $\Phi : G \to H$ such that $\Phi_*(e) = \phi$.*

Proof. a) Let $X \in \mathfrak{g}$, and let $\widetilde{X}$ be the left-invariant vector field on G with $\widetilde{X}(e) = X$. We define the curve $g_X(t)$ as the unique solution to the equation

$$\dot{g}_X(t) = \widetilde{X}(g_X(t))$$

with the initial condition $g_X(0) = e$. The two curves with parameter s,

$$\phi_1(s) = g_X(t+s) \qquad \text{and} \qquad \phi_2(s) = g_X(t)g_X(s),$$

both satisfy the equation $\frac{d}{ds}\phi(s) = \widetilde{X}(\phi(s))$ and the initial condition $\phi(0) = g_X(t)$. Hence, they coincide and we get $g_X(t)g_X(s) = g_X(t+s)$.

b) The image under Φ of a 1-parameter subgroup $\{g_X(t)\}$ is a 1-parameter subgroup $h(t)$ in H with $\dot{h}(0) = Y := \Phi_*(e)X$. Hence, $\Phi(g_X(t)) = h_Y(t)$ and $\Phi(\exp X) = \exp Y = \exp \Phi_*(e)X$.

The proof of c) is more involved and we omit it. Note only that for non-simply connected G the statement can be wrong (consider the case $G = H = \mathbf{T}^1$). □

In particular, we have the following important formula, relating Ad and ad:

$$\mathrm{Ad}\,(\exp X) = e^{\mathrm{ad}\,X}. \tag{12}$$

For matrix groups the exponential map is given by the usual formula for an exponent:

$$\exp X = e^X = \sum_{n \geq 0} \frac{X^n}{n!}$$

and (12) gives the following identity:

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + \frac{1}{6}[X, [X, [X, Y]]] + \cdots.$$

The remarkable discovery of Sophus Lie, the founder of Lie group theory, was that a Lie group can be almost uniquely reconstructed from its Lie algebra. To formulate the precise result, we introduce the following definition.

Two Lie groups G_1 and G_2 are called **locally isomorphic** if there exists a diffeomorphism φ between two neighborhoods of unity $U_i \subset G_i$, $i = 1, 2$, which is compatible with the multiplication laws: $\varphi(xy) = \varphi(x)\varphi(y)$ whenever both sides make sense.

Theorem 3. a) *For any real Lie algebra* $\mathfrak{g}$ *there exists a Lie group* G *such that* $\mathrm{Lie}\,(G) = \mathfrak{g}$ *and this group is unique up to local isomorphism.*

b) *Among all connected Lie groups* G *such that* $\mathrm{Lie}(G) = \mathfrak{g}$ *there exists exactly one (up to isomorphism) simply connected Lie group* $\widetilde{G}$.

c) *Let* C *be the center of* $\widetilde{G}$. *Every connected Lie group* G *with* $\mathrm{Lie}(G) = \mathfrak{g}$ *is isomorphic to* $\widetilde{G}/\Gamma$ *where* Γ *is a discrete subgroup of* $\widetilde{G}$ *contained in* C.

Scheme of the proof. Sophus Lie proved a) using the partial differential equations for a 1-parameter subgroup in G. One can also use the Ado theorem to prove the existence of G. The uniqueness and claim b) follow from Theorem 2 if we observe that any Lie group G admits a simply connected covering group $\widetilde{G}$ and the natural projection $p : \widetilde{G} \to G$ is a local isomorphism, so that $\Gamma = \ker p$ is a discrete normal subgroup. Finally, c) follows from

Lemma 2. *Let* G *be a connected Lie group, and let* Γ *be a normal discrete subgroup. Then* Γ *is contained in the center of* G.

Proof of the lemma. Consider the map $A_\gamma : G \to \Gamma : g \mapsto g\gamma g^{-1}$. This map is continuous, hence the image is a connected set. But Γ is discrete and the connected subsets are just points. Therefore $A_\gamma(G) = A_\gamma(e) = \{\gamma\}$. Hence, $g\gamma g^{-1} = \gamma$ for all $g \in G$ and $\gamma \in \Gamma$. $\square$

Example 2. Let $\mathfrak{g} = \mathbb{R}^n$ be the n-dimensional Lie algebra with the trivial (zero) commutator. Then the corresponding simply connected Lie group $\widetilde{G}$ is just $\mathbb{R}^n$ with the group law given by vector addition. This group is abelian, hence coincides with its center.

The classical **Kronecker Theorem** claims that every discrete subgroup Γ in G has the form

$$\mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_k$$

where $v_1, v_2, \ldots, v_k$ are linearly independent vectors.

Therefore, there are exactly $n+1$ non-isomorphic connected Lie groups G for which $\mathrm{Lie}(G) = \mathbb{R}^n$, namely $\mathbb{T}^k \times \mathbb{R}^{n-k}$, $0 \leq k \leq n$. $\diamondsuit$

A very important fact is the following hereditary property of Lie groups.

Theorem 4. a) *Any closed subgroup H in a Lie group G is a smooth submanifold and a Lie subgroup in G.*

b) *For any closed subgroup H in a Lie group G the coset space G/H admits the unique structure of a smooth manifold compatible with the group action.* $\square$

The non-formal meaning of this theorem is that practically all reasonable groups are Lie groups. This result should not be overestimated. For example, all discrete groups are 0-dimensional Lie groups. But it does not give us any useful information.

On the contrary, for a connected group G the fact that G is a Lie group is very important and allows us to reduce many geometric, analytic, or topological problems about G to pure algebraic questions about $\mathfrak{g}$.

In conclusion we formulate one more useful observation here.

Lemma 3. *Let G be a Lie group, and let H be an open subgroup. Then H is also closed in G.*

Proof of the lemma. All cosets gH are open in G. Therefore, the set $\bigcup_{g \notin H} gH$ is open. Hence, its complement H is closed. $\square$

Corollary. *A connected Lie group G is generated by any neighborhood U of the unit element.*

Proof of the corollary. First, replacing U, if necessary, by a smaller neighborhood $V = U \cap U^{-1}$, we can assume that $U = U^{-1}$. Then the subgroup $H \subset G$ generated by U is the union of open sets

$$U^1 := U, \quad U^2 := U \cdot U, \quad U^3 := U \cdot U \cdot U, \quad \text{etc.},$$

and, therefore, is open. By Lemma 3 it is also closed, hence coincides with G. □

1.4. Universal enveloping algebras.

A Lie algebra is a more algebraic notion than a Lie group, but still Lie algebras are not as clear and customary objects as associative algebras. Therefore, it is important to know that from the representation-theoretic point of view one can replace a Lie algebra $\mathfrak{g}$ by a certain associative algebra $U(\mathfrak{g})$, which has exactly the same category of representations.

The formal definition of $U(\mathfrak{g})$ is better formulated in categorical language. Consider the category $\mathcal{C}(\mathfrak{g})$ whose objects are linear maps of $\mathfrak{g}$ into some complex associative unital[7] algebra A (its own for each object), satisfying the condition

$$\varphi([X, Y]) = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X). \tag{13}$$

A morphism from $(\varphi : \mathfrak{g} \to A)$ to $(\psi : \mathfrak{g} \to B)$ is by definition a morphism $\alpha : A \to B$ of unital associative algebras such that the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{g} & = & \mathfrak{g} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ A & \xrightarrow{\alpha} & B. \end{array}$$

Theorem 5. *The category $\mathcal{C}(\mathfrak{g})$ has a universal (initial) object that is denoted by $(i : \mathfrak{g} \to U(\mathfrak{g}))$. The algebra $U(g)$ is called the* **universal enveloping algebra** *for $\mathfrak{g}$; the morphism $i : \mathfrak{g} \to U(g)$ is called the* **canonical embedding** *of $\mathfrak{g}$ into $U(g)$.* □

This definition, in spite of its abstractness, is rather convenient. For example, it implies immediately that to any representation π of $\mathfrak{g}$ in a vector space V there corresponds a representation of $U(\mathfrak{g})$ in the same space. Moreover, the categories of $\mathfrak{g}$-modules and $U(\mathfrak{g})$-modules are canonically isomorphic. (Check it!)

[7] A unital algebra is an algebra with a unit. In the category of unital algebras morphisms are so-called unital homomorphisms that send units to units.

On the other hand, the definition above is not constructive. Therefore, we give two other definitions of $U(g)$, which are more "down-to-earth". One can use these definitions to prove Theorem 5.

First, choose a basis $X_1, \ldots, X_n$ in $\mathfrak{g}$ and consider the unital associative algebra $A(\mathfrak{g})$ over $\mathbb{C}$ generated by $X_1, \ldots, X_n$ with relations

$$X_i X_j - X_j X_i - [X_i, X_j] = 0. \tag{14}$$

It means, by definition, that $A(\mathfrak{g})$ is a quotient algebra of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal generated by the terms occurring in (14).

Second, let G be any connected Lie group with $\operatorname{Lie}(G) = \mathfrak{g}$. Consider the algebra $B(\mathfrak{g})$ of all differential operators on G that are invariant under left shifts on G. The Lie algebra $\mathfrak{g}$ is embedded in $B(\mathfrak{g})$ via the left-invariant vector fields (= first order differential operators without constant term): $X \mapsto \widetilde{X}$ (see Appendix II.2.1).

Exercise 2. Show that the algebras $A(\mathfrak{g})$ and $B(\mathfrak{g})$ defined above are isomorphic to $U(\mathfrak{g})$.

Hint. Use the universal property of $U(\mathfrak{g})$ to define the maps $\alpha : U(\mathfrak{g}) \to A(\mathfrak{g})$ and $\beta : U(\mathfrak{g}) \to B(\mathfrak{g})$ and prove that these maps are surjective.

To prove that α and β are injective, use the famous

Poincaré-Birkhoff-Witt Theorem. *Let $X_1, \ldots, X_n$ be any basis in $\mathfrak{g}$. Then the monomials $M^k := i(X_1)^{k_1} \cdots i(X_n)^{k_n}$, $k := (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, form a basis of the vector space $U(\mathfrak{g})$.* □

So, any choice of a basis in $\mathfrak{g}$ allows us to identify the space $U(\mathfrak{g})$ with the space $S(\mathfrak{g})$ of ordinary polynomials in $X_1, \ldots, X_n$. Namely,

$$\sum_{k \in \mathbb{Z}_+^n} c_k M^k \quad \longleftrightarrow \quad \sum_{k \in \mathbb{Z}_+^n} c_k X^k \qquad \text{where } X^k := X_1^{k_1} \cdots X_n^{k_n}.$$

A more natural and more convenient identification of $U(\mathfrak{g})$ and $S(\mathfrak{g})$ is given by the so-called **symmetrization map**. This map $\mathbf{sym} : S(\mathfrak{g}) \to U(\mathfrak{g})$ is defined by the formula

$$\mathbf{sym}(P) = P\big(\partial_{\alpha_1}, \ldots, \partial_{\alpha_n}\big) e^{\alpha_1 i(X_1) + \cdots + \alpha_n i(X_n)} \Big|_{\alpha_1 = \cdots = \alpha_n = 0}. \tag{15}$$

The name "symmetrization map" comes from the following property of **sym**.

Exercise 3. Show that for any elements $Y_1, \ldots, Y_m$ from $\mathfrak{g}$ we have

$$\mathbf{sym}\,(Y_1 Y_2 \cdots Y_m) = \frac{1}{m!} \sum_{s \in S_m} i\big(Y_{s(1)}\big) i\big(Y_{s(2)}\big) \cdots i\big(Y_{s(m)}\big). \qquad ♣$$

In particular, **sym** satisfies the equation

$$(16) \qquad \mathbf{sym}(Y^k) = \big(i(Y)\big)^k \qquad \text{for all } Y \in \mathfrak{g},\ k \in \mathbb{Z}_+,$$

and is uniquely determined by it. We refer to Chapter 2, Section 1.1.2, for the worked example of this map.

Recall that there is a natural G-action on $S(\mathfrak{g})$ and on $U(g)$ coming from the adjoint representation. The important advantage of **sym** is that it is an intertwiner for this G-action. In matrix notation this property has the form

$$(17) \qquad \mathbf{sym}\big(P(g \cdot X \cdot g^{-1})\big) = g \cdot \mathbf{sym}\big(P(X)\big) \cdot g^{-1}$$

where the elements of $U(\mathfrak{g})$ are interpreted as differential operators on G.

The proof follows immediately from (16), if we take into account that the space $S^k(\mathfrak{g})$ of homogeneous polynomials of degree k is spanned by $\{Y^k,\ Y \in \mathfrak{g}\}$.

Let $Z(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$, and let $Y(\mathfrak{g})$ be the algebra of G-invariant elements of $S(\mathfrak{g})$ that can also be considered as polynomial functions on $\mathfrak{g}^*$, constant along coadjoint orbits.

Gelfand–Harish-Chandra Theorem. *The map* **sym** *defines a bijection of* $Y(\mathfrak{g})$ *onto* $Z(\mathfrak{g})$.

Proof. From (17) we see that $\mathbf{sym}(Y)$ is the set of G-invariant elements of $U(\mathfrak{g})$. Since G is connected, the G-invariance is equivalent to $\mathfrak{g}$-invariance. Therefore, $\mathbf{sym}(Y)$ consists of elements A satisfying $[X, A] = 0$ for all $X \in \mathfrak{g}$. But this is exactly the center of $U(\mathfrak{g})$. □

Warning. Both $Y(\mathfrak{g})$ and $Z(\mathfrak{g})$ are commutative algebras. Moreover, we shall see later that for any Lie algebra $\mathfrak{g}$ they are isomorphic. But the map **sym** in general is not an algebra homomorphism. It becomes an isomorphism after the appropriate correction (see the modified Rule 7 in Chapter 4). ♠

2. Review of the set of Lie algebras

Here we try to give the reader an overview of the totality of Lie algebras.

2.1. Sources of Lie algebras.

The main source and *raison d'être* of real Lie algebras is the theory of Lie groups (see Section 1.3). There are, however, at least two other sources of pure algebraic nature.

Lemma 4. a) *Let A be an arbitrary (not necessarily associative) algebra, and let* $\operatorname{Der} A$ *denote the space of all derivations of A, i.e. linear operators $D: A \to A$ satisfying the Leibnitz rule*

$$D(ab) = D(a)b + aD(b). \tag{18}$$

Then $\operatorname{Der} A$ *is a Lie algebra with respect to the commutator*

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1. \tag{19}$$

b) *Let A be an associative algebra; introduce the commutator in A by*

$$[a, b] = ab - ba. \tag{20}$$

Then the space A with this operation is a Lie algebra. Sometimes, it is denoted by A^{Lie} or $\operatorname{Lie}(A)$.

The proof is a check of the Jacobi identity by a direct computation which is the same in both cases. □

One can deduce a more conceptual proof from the fact that the Lie algebras defined in Lemma 4 are related to certain Lie groups, possibly infinite-dimensional.

Remark 1. Both constructions of Lie algebras given by Lemma 4 admit a natural generalization. To formulate it, we have to assume that A is a $\mathbb{Z}_2$-graded algebra, consider the graded derivations, and correct the definition of the commutator according to the following **sign rule**:

If a formula contains a product **ab** *of two homogeneous elements, then each time these elements occur in the inverse order* **ba**, *the sign* $(-1)^{\deg \mathbf{a} \cdot \deg \mathbf{b}}$ *must be inserted.*

So, equations (2), (3), (22), (23), (24) take the form

$$\operatorname{ad} X\,([Y, Z]) = [\operatorname{ad} X\,(Y), Z] + (-1)^{\deg X \cdot \deg Y}[Y, \operatorname{ad} X\,(Z)], \tag{2'}$$

$$\operatorname{ad}[X, Y] = \operatorname{ad} X \circ \operatorname{ad} Y - (-1)^{\deg X \cdot \deg Y} \operatorname{ad} Y \circ \operatorname{ad} X, \tag{3'}$$

$$D(ab) = D(a)b + (-1)^{\deg D \cdot \deg a} a D(b), \tag{18'}$$

$$[D_1, D_2] = D_1 D_2 - (-1)^{\deg D_1 \cdot \deg D_2} D_2 D_1, \tag{19'}$$

$$[a, b] = ab - (-1)^{\deg a \cdot \deg b} ba. \tag{20'}$$

The new object arising in this way is called a **Lie superalgebra**. Many important Lie algebras can be considered as even parts of some superalgebras. This approach turns out to be very useful in representation theory and other applications.

For readers who do not want to be overloaded by superterminology, we give the description of Lie superalgebras in terms of ordinary Lie algebras and their modules. Namely, a Lie superalgebra is the collection of the following data:

1. An ordinary Lie algebra $\mathfrak{g}_0$.
2. A $\mathfrak{g}_0$-module $\mathfrak{g}_1$.
3. A symmetric pairing $[\cdot,\,\cdot]$ of $\mathfrak{g}_0$-modules: $\mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_0$.

This data should satisfy the condition:

$$(*) \qquad X^3 := [X,\, X] \cdot X = 0 \qquad \text{for any } X \in \mathfrak{g}_1.$$

In fact, points 1, 2, 3 and equation (*) correspond to four different variants of the super-Jacobi identity which can involve 0, 1, 2, or 3 odd elements.

Example 3. Let $\mathfrak{g}_0 = \mathfrak{sl}(2,\,\mathbb{R}),\ \mathfrak{g}_1 = \mathbb{R}^2$.

The action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is the usual multiplication of a matrix and a column vector. The pairing is

$$[x,\, y] = (xy^t + yx^t)J_2 = \begin{pmatrix} -x_1y_2 - x_2y_1 & -2x_2y_2 \\ 2x_1y_1 & x_1y_2 + x_2y_1 \end{pmatrix}.$$

Condition (*) holds because $[x,\, x]x = 2xx^tJ_2x = 0$.

This Lie superalgebra is denoted by $\mathfrak{osp}(1\,|\,2,\,\mathbb{R})$ and is the simplest representative of a series of superalgebras $\mathfrak{osp}(k\,|\,2n,\,\mathbb{R})$ whose even parts are direct sums $\mathfrak{so}(k,\,\mathbb{R}) \oplus \mathfrak{sp}(2n,\,\mathbb{R})$, while the odd parts are tensor products $\mathbb{R}^k \otimes \mathbb{R}^{2n}$. ◊

Two particular cases of Lemma 4 deserve more detailed discussion.

1. Let M be a smooth manifold. Put $A = \mathcal{A}(M)$ in Lemma 4 a). As we have seen in Appendix II.2.1, derivations of $\mathcal{A}(M)$ correspond to smooth vector fields on M. So, we obtain a Lie algebra structure on the infinite-dimensional space *Vect* M. The commutator in this case is the Lie bracket of vector fields described in Appendix II.2.3.

In terms of local coordinates this operation has the form

$$[v,\, w]^i = v^j\partial_j w^i - w^j\partial_j v^i.$$

One of the main results of the original Sophus Lie theory can be formulated in modern terms as the

Sophus Lie Theorem. *Every n-dimensional real Lie algebra $\mathfrak{g}$ can be realized as a subalgebra of Vect M for some manifold M of dimension n.* □

So, this example has a universal character.

2. Put $\mathcal{A} = \mathrm{Mat}_n(\mathbb{R})$ in Lemma 4 b). The Lie $(\mathcal{A})$ in this case is usually denoted by $\mathfrak{gl}(n, \mathbb{R})$. As a vector space it coincides with $\mathrm{Mat}_n(\mathbb{R})$, but the basic operation is the commutator (20) instead of matrix multiplication in $\mathrm{Mat}_n(\mathbb{R})$.

This example is also universal as we saw before (see Ado's Theorem).

Before going further, let us introduce some terminology. The collection of all Lie algebras over a given field K forms a category $\mathcal{LA}(K)$ where morphisms are **Lie algebra homomorphisms**, i.e. linear maps $\pi : \mathfrak{g}_1 \to \mathfrak{g}_2$ that preserve the brackets:

$$[\pi(X), \pi(Y)] = \pi([X, Y]). \tag{21}$$

Homomorphisms of a real Lie algebra $\mathfrak{g}$ to $\mathfrak{gl}(n, \mathbb{R})$ (resp. $\mathfrak{gl}(n, \mathbb{C})$) are called real (resp. complex) **linear representations** of $\mathfrak{g}$. We can identify $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) with any real (resp. complex) vector space V. Then we use the notation $\mathfrak{gl}(V)$ and speak of representations of $\mathfrak{g}$ in the space V, which is called a representation space or **$\mathfrak{g}$-module**.

The collection of all $\mathfrak{g}$-modules forms a category $\mathfrak{g}$-$\mathcal{M}od$ where morphisms are linear maps that commute with the $\mathfrak{g}$-action. They are called **intertwining operators** or simply **intertwiners**.

We shall consider in this book not only finite-dimensional but also infinite-dimensional representations of Lie algebras.

2.2. The variety of structure constants.

To define a Lie algebra structure on a vector space V one usually chooses a basis $X_1, \ldots, X_n$ and defines the structure constants c_{ij}^k by

$$[X_i, X_j] = c_{ij}^k X_k.$$

These constants satisfy two sets of equations: the linear equations

$$c_{ij}^k = -c_{ji}^k \qquad \text{(Antisymmetry)}; \tag{22}$$

and the quadratic equations:

$$\underset{ijk}{\circlearrowleft} c_{ij}^s c_{sk}^m = 0 \qquad \text{(Jacobi identity)}. \tag{23}$$

Thus, the collection of all real Lie algebras of dimension n with a fixed basis can be viewed as the set $A_n(\mathbb{R})$ of real points of the affine algebraic manifold A_n defined by the equations (22), (23).

If we change the basis, the structure constants will change according to the standard action of $GL(n, \mathbb{R})$ on tensors of type (2, 1). So, two points of $A_n(\mathbb{R})$ correspond to isomorphic Lie algebras iff they belong to the same $GL(n, \mathbb{R})$-orbit.

We see that the collection $L_n(\mathbb{R})$ of isomorphism classes of n-dimensional real Lie algebras is just the set $A_n(\mathbb{R})/GL(n, \mathbb{R})$ of $GL(n, \mathbb{R})$-orbits in $A_n(\mathbb{R})$.

Note also that the orbit corresponding to a Lie algebra $\mathfrak{g}$, as a homogeneous space, has the form $GL(n, \mathbb{R})/\operatorname{Aut}\mathfrak{g}$ where $\operatorname{Aut}\mathfrak{g}$ is the group of automorphisms of the Lie algebra $\mathfrak{g}$. So, Lie algebras with a bigger automorphism group correspond to smaller orbits.

A **deformation** of an n-dimensional Lie algebra $\mathfrak{g}$ is a curve $l(t)$, $0 \leq t < \epsilon$, in $L_n(\mathbb{R})$ that is a projection $l(t) = p(a(t))$ of a smooth curve $a(t)$ in $A_n(\mathbb{R})$ such that $l(0)$ represents the class of $\mathfrak{g}$.

Two deformations $l_1(t) = p(a_1(t))$ and $l_2(t) = p(a_2(t))$ are **equivalent** if there is a smooth function ϕ mapping $[0, \epsilon_1)$ to $[0, \epsilon_2)$ with $\phi(0) = 0$ and a smooth curve $g(t)$ in $GL(n, \mathbb{R})$ such that $a_1(t) = g(t) \cdot a_2(\phi(t))$ in some neighborhood of 0.

If an orbit O is open in A_n, we say that the corresponding Lie algebra $\mathfrak{g}$ is **rigid**. In this case all deformations of $\mathfrak{g}$ are **trivial**, i.e. equivalent to the constant deformation $l_0(t) \equiv l(0)$.

In other words, $\mathfrak{g}$ is rigid if a small perturbation of structure constants leads to an isomorphic Lie algebra.

Proposition 2. *All semisimple Lie algebras are rigid.* □

The converse is not true; e.g. the only non-commutative 2-dimensional Lie algebra $\mathfrak{aff}(1, \mathbb{R})$ is rigid but solvable.

If an orbit O has non-empty boundary, then this boundary is a union of smaller orbits. The Lie algebras corresponding to these orbits are called **contractions** of the initial Lie algebra. In particular, the abelian Lie algebra $\mathbb{R}^n$ is a contraction of any n-dimensional Lie algebra.

To illustrate the introduced notions, we consider in more detail the sets $A_n(\mathbb{R})$ and $L_n(\mathbb{R})$ for small n.

We can use the linear equations (22) to simplify the system and reduce it to a system of $\frac{n^2(n-1)(n-2)}{6}$ quadratic equations in $\frac{n^2(n-1)}{2}$ unknowns. But the full description of all solutions is still very difficult for large n.

A complete classification of Lie algebras has been obtained up until now only for $n \leq 8$ (using a completely different approach).

On the other hand, for small n it is not difficult to solve the system (22), (23) directly and thus describe all Lie algebras of small dimensions.

Exercise 4. Show that:

a) There is only one 1-dimensional real Lie algebra: the real line $\mathbb{R}$ with zero commutator.

b) There are exactly two non-isomorphic 2-dimensional real Lie algebras: the trivial Lie algebra $\mathbb{R}^2$ with zero commutator and the rigid Lie algebra $\mathrm{aff}(1, \mathbb{R})$ with the commutation relation $[X, Y] = Y$.

Hint. For $n \leq 2$ the quadratic equations (23) hold automatically. ♣

Consider now 3-dimensional real Lie algebras. For $n = 3$ the system (22), (23) admits a simple geometric description. Let us replace the tensor c_{ij}^k of type (2, 1) by a tensor density $b^{kl} := c_{ij}^k \epsilon^{ij\,l}$ where $\epsilon^{ij\,l}$ is the standard 3-vector in $\mathbb{R}^3$.

This new quantity b^{kl} splits into symmetric and antisymmetric parts:

$$b^{kl} = s^{kl} + a^{kl}, \qquad s^{kl} = s^{lk}, \quad a^{kl} = -a^{lk},$$

which are transformed separately under the action of $GL(3, \mathbb{R})$.

Further, we replace the antisymmetric tensor a^{kl} by the covector $v_i := \epsilon_{ikl} a^{kl}$. Using the identities

$$\epsilon_{ijk}\epsilon^{jkl} = 2\delta_k^l, \qquad \epsilon_{ijs}\epsilon^{skl} = \delta_i^k \delta_j^l - \delta_i^l \delta_j^k,$$

we can reconstruct the initial tensor c_{ij}^k from s^{kl} and v_i as

$$c_{ij}^k = \frac{1}{2} s^{kl} \epsilon_{ij\,l} + \frac{1}{4}(\delta_j^k v_i - \delta_i^k v_j).$$

It turns out that the system (22), (23) in terms of new quantities s and v takes a very simple form

$$s^{kl} v_l = 0. \tag{24}$$

We see from (24) that the manifold A_3 is reducible and splits into two irreducible components:

1) the linear component defined by $v = 0$ (in the initial terms $c_{ik}^k = 0$);

2) the non-linear component given by conditions: $\det s = 0, \ v \in \ker s$.

Both components are 6-dimensional affine varieties; their intersection is non-empty and has dimension 5.

The action of $g \in GL(3, \mathbb{R})$ on $A_3(\mathbb{R})$ in new coordinates (v, s) has the form:

$$v \mapsto (g^t)^{-1} v, \qquad s \mapsto g s g^t \cdot \det g^{-1}. \tag{25}$$

In other words, v^t transforms as a covector (row vector) and s as a product of a quadratic form on $(\mathbb{R}^3)^*$ by a volume form on $\mathbb{R}^3$.

The first component consists of quadratic forms $s = s^{ij}$ on $\mathbb{R}^3$ on which the group $GL(3, \mathbb{R})$ acts according to the second part of (25). The only invariant of this action is the signature[8] (n_+, n_0, n_-) considered up to equivalence $(a, b, c) \sim (c, b, a)$.

Thus, we get the following list:

Lie algebra	signature	dimension of the orbit
$\mathfrak{su}(2) \cong \mathfrak{so}(3)$	$(3, 0, 0) \sim (0, 0, 3)$	6
$\mathfrak{sl}(2) \cong \mathfrak{so}(2, 1)$	$(2, 0, 1) \sim (1, 0, 2)$	6
$\mathfrak{so}(2) \ltimes \mathbb{R}^2$	$(2, 1, 0) \sim (0, 1, 2)$	5
$\mathfrak{so}(1, 1) \ltimes \mathbb{R}^2$	$(1, 1, 1)$	5
Heisenberg algebra $\mathfrak{h}$	$(1, 2, 0) \sim (0, 2, 1)$	4
trivial Lie algebra $\mathbb{R}^3$	$(0, 3, 0)$	0

We see that the first component of $L_3(\mathbb{R})$ consists of six points, two of which are open. They correspond to the rigid Lie algebras $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ and $\mathfrak{sl}(2) \cong \mathfrak{so}(2, 1)$.

Consider the Lie algebras corresponding to the second component. In this case we have a non-zero vector $v = \{v_m\}$ such that $s^{km} v_m = 0$. Besides, $a^{km} v_m = \frac{1}{2}\epsilon^{kmp} v_p v_m = 0$. Therefore, $b^{km} v_m = 0$ and $c_{ij}^k v_k = 0$.

This means that $[\mathfrak{g}, \mathfrak{g}] \subset v^\perp$. So, our Lie algebra has a 2-dimensional ideal. It turns out that all such Lie algebras have the form of a semidirect product $\mathbb{R} \ltimes \mathbb{R}^2$. This means that in the appropriate basis the commutation relations have the form

$$[X, Y] = \alpha Y + \beta Z, \qquad [X, Z] = \gamma Y + \delta Z, \qquad [Y, Z] = 0.$$

The matrices

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$$

define isomorphic Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ iff

$$A' = c \cdot BAB^{-1} \qquad \text{for some } c \neq 0 \text{ and } B \in GL(2, \mathbb{R}). \tag{26}$$

[8] For a quadratic form s on $\mathbb{R}^n$ the numbers n_+, n_0, and n_- denote respectively the number of positive, zero, and negative coefficients c_i in the canonical representation $s(x) = \sum_{i=1}^n c_i x_i^2$. The triple (n_+, n_0, n_-) is called the **signature** of s.

Let λ, μ be the eigenvalues of A; then A' has eigenvalues $(c\lambda, c\mu)$. Also, we cannot distinguish the pairs (λ, μ) and (μ, λ). Thus, the quantity

$$f(A) = \frac{\mu}{\lambda} + \frac{\lambda}{\mu} = \frac{\operatorname{tr}(A^2)}{\det A}$$

is the simplest invariant of the transformations (26). It is a rational function of four variables α, β, γ, δ that is correctly defined when $(\lambda, \mu) \neq (0, 0)$ (i.e. outside the cone given by $\operatorname{tr} A = \det A = 0$).

Warning. A rational function f of n variables can be written in the form $\frac{P}{Q}$ where P and Q are relatively prime polynomials. We consider this expression as a map from $\mathbb{C}^n$ to $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. For $n = 1$ this map is everywhere defined and continuous. For $n \geq 2$ the map f is defined and continuous only outside the so-called indeterminancy set $I(f)$ of codimension 2 given by the equations $P = Q = 0$. ♠

To express $f(A)$ in terms of structure constants, let us consider the bilinear forms s and a, which are defined by tensors a^{kl} and s^{kl} on $\mathfrak{g}$. In a generic point they have a 1-dimensional kernel spanned by the vector v. The induced forms on the quotient space $\mathfrak{g}/\mathbb{R}v$ are non-degenerate and the ratio of their discriminants equals

$$r(A) = -\left(\frac{\lambda+\mu}{\lambda-\mu}\right)^2 = \frac{2+f(A)}{2-f(A)}.$$

So, the second component admits a non-trivial $GL(3, \mathbb{R})$-invariant rational function $f(A)$. Therefore, it splits into a 1-parameter family of invariant 5-dimensional levels $f(A) = c$ and a singular 4-dimensional subset $\lambda = \mu = 0$ where $f(A)$ is not defined. Most of these levels are single $GL(3, \mathbb{R})$-orbits. The levels $f(A) = \pm 2$ and the singular set split into two orbits each.

We collect this information in the following table:

$c = f(A)$	number of orbits	eigenvalues of $\operatorname{ad} X$
$-\infty < c < -2$	1	real, of different signs;
$c = -2$	2	$\lambda = -\mu \neq 0$, real or pure imaginary;
$-2 < c < 2$	1	complex conjugate;
$c = 2$	2	real, equal, non-zero;
$c > 2$	1	real, different, of the same sign;
$c = \infty$	1	one zero eigenvalue;
not defined	2	two zero eigenvalues.

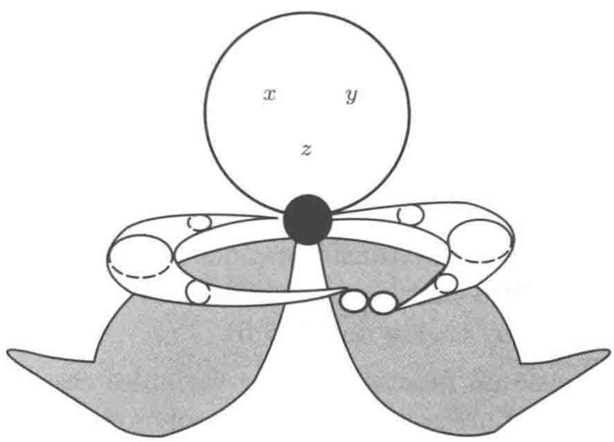

Figure 1

Note that the intersection of the two components contains the level $f(A) = -2$ and the singular set. We see that the Heisenberg algebra is situated in the core of the set $L_3(\mathbb{R})$ of all 3-dimensional Lie algebras.

The final structure of $A_3(\mathbb{R})/GL(3, \mathbb{R})$ is shown in Figure 1.

In conclusion we bring some information about the manifold A_n for $n > 3$ (see [**KN**], [**Ner2**]). It is known that

A_4 has four irreducible components, all of dimension 12;

A_5 has seven components of dimensions 21, 20, 20, 20, 20, 20, 19;

A_6 has 17 components of dimensions 32, 31, 30, 30, ... , 30.

It is also known that dim A_n is asymptotically equal to $\frac{2}{27}n^3$. The most "massive" components are related to nilpotent Lie algebras of type $\mathbb{R}^k \ltimes \mathbb{R}^l$ with $k \approx 2l$.

The number of irreducible components of A_n grows still faster, at least as fast as the number $p(n)$ of partitions of n, which is asymptotically equal to $\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{\frac{2}{3}n}}$.

Since different irreducible components behave as independent algebraic manifolds, there is no such thing as a "generic" Lie algebra of given dimension.

Fortunately, one can interpret A_n as the set of linear Poisson structures on $\mathbb{R}^n$ (see Appendix II.3.2). In this way A_n is embedded in the infinite-dimensional connected manifold of all (non-linear) Poisson structures on $\mathbb{R}^n$. This approach was recently used by Kontsevich to prove the general quantization theorem for Poisson manifolds.

2.3. Types of Lie algebras.

The direct study of equations (22), (23) is already rather difficult for $n = 4$.

There is another approach to the classification problem for Lie algebras. It is based on the study of their inner structure. For the reader's convenience we recall below some standard definitions.

Definition 1. A Lie algebra $\mathfrak{g}$ is an **extension** of a Lie algebra $\mathfrak{g}_1$ by a Lie algebra $\mathfrak{g}_2$ if the following exact sequence exists:

$$0 \longrightarrow \mathfrak{g}_2 \xrightarrow{i} \mathfrak{g} \xrightarrow{p} \mathfrak{g}_1 \longrightarrow 0. \tag{27}$$

Here all maps are Lie algebra homomorphisms and the exactness means that the image of each arrow coincides with the kernel of the next one.

In other words, (27) means that $\mathfrak{g}$ contains an ideal isomorphic to $\mathfrak{g}_2$ and the quotient Lie algebra $\mathfrak{g}/\mathfrak{g}_2$ is isomorphic to $\mathfrak{g}_1$.

The extension is called **central** if $i(\mathfrak{g}_2)$ is contained in the center of $\mathfrak{g}$.

The extension is called **trivial** if the map p in (27) admits a section, i.e. a homomorphism $s : \mathfrak{g}_1 \to \mathfrak{g}$ such that $p \circ s = \mathrm{Id}$. In this case we also say that $\mathfrak{g}$ is a **semidirect product** of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. It is denoted by $\mathfrak{g}_1 \ltimes \mathfrak{g}_2$.

We now can introduce several important types of Lie algebras:

1) **Commutative**, or **abelian** Lie algebras – those with zero commutator.

2) The class of **solvable** Lie algebras – the minimal collection of Lie algebras that contains all abelian Lie algebras and is closed under extensions.

3) The class of **nilpotent** Lie algebras – the minimal collection of Lie algebras that contains all abelian Lie algebras and is closed under central extensions.

4) The class of **semisimple** Lie algebras – the minimal collection of Lie algebras that contains all non-abelian simple[9] Lie algebras and is closed under extensions.

For your information we formulate several facts here from real Lie algebra theory.

Levi's Theorem. *Any Lie algebra $\mathfrak{g}$ has a unique maximal solvable ideal $\mathfrak{r}$, the corresponding quotient Lie algebra $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$ is semisimple, and $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$.* □

[9]A Lie algebra is called simple if it has no **proper** ideals, i.e. different from $\{0\}$ and from the whole Lie algebra.

Cartan's Theorem. *Any semisimple Lie algebra is isomorphic to a direct sum of non-abelian simple Lie algebras.* □

So, the classification of all Lie algebras is reduced to three problems:

A) Describe all simple Lie algebras.

B) Describe all solvable Lie algebras.

C) Describe all semidirect products $\mathfrak{g}_1 \ltimes \mathfrak{g}_2$ where $\mathfrak{g}_1$ is semisimple and $\mathfrak{g}_2$ is solvable.

At the present time only problem A) is completely solved (see the next sections), while B) and C) are regarded as hopeless.

3. Semisimple Lie algebras

The class of semisimple Lie algebras is the most interesting for many applications and therefore has been the most thoroughly studied. The structure of these Lie algebras and their classification are related to the notion of a root system.

We include the general facts about root systems in the next sections. We hope that this allows our readers not only to get the general impression about this theory but also use the basic results in their research. For detailed proofs and further information we refer to the books [**Bou**], [**FH**], [**Hu**], [**MPR**], and [**OV**].

3.1. Abstract root systems.

We discuss here the notion and properties of a remarkable geometric object: a root system in Euclidean space. It appears in surprisingly many quite different domains of mathematics (see, e.g., [**H²SV**]). Our main goal is to list basic facts and explain how to use them. So, the detailed proofs are given only if they are not too involved and help in understanding. We refer to [**Bou**], [**FH**], and [**OV**] for further information.

Definition 2. A finite set $R \subset \mathbb{R}^n$ is called a **root system** if it satisfies two conditions:

R1. $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in R$;

R2. $\beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in R$ for all $\alpha, \beta \in R$.

The elements of R are called **roots**.

Besides these main conditions, some additional properties are often required that define special kinds of root systems. We formulate these requirements here together with the name of the corresponding kind of root system.

R3a. The set R spans the whole $\mathbb{R}^n$ (Non-degenerate root system).

R3b. The set R cannot be represented in the form $R_1 \sqcup R_2$ with $R_1 \perp R_2$ (Indecomposable root system).

R3c. If $\alpha \in R$, then $2\alpha \notin R$ (Reduced root system).

R3d. All vectors from R have the same length (Simply-laced root system).

Let us comment on the main definition and the additional requirements.

1. The geometric meaning of R1: the angles between two roots can be only from the following list:

$$0, \quad \frac{\pi}{6}, \quad \frac{\pi}{4}, \quad \frac{\pi}{3}, \quad \frac{\pi}{2}, \quad \frac{2\pi}{3}, \quad \frac{3\pi}{4}, \quad \frac{5\pi}{6}, \quad \pi,$$

and for any two non-perpendicular roots the ratio of the squares of their lengths can be only 1, 2, or 3, depending on the angle between them. All possible configurations of two non-perpendicular roots are shown in Figure 2.

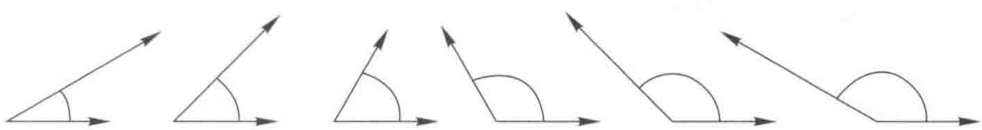

Figure 2

2. Let $M_\alpha \subset \mathbb{R}^n$ be the hyperplane orthogonal to $\alpha \in R$. It is called a **mirror** corresponding to α and the reflection with respect to this mirror is denoted by s_α. The condition R2 means that the set R is symmetric with respect to all mirrors, i.e. is invariant under all reflections s_α, $\alpha \in R$.

The group generated by reflections s_α, $\alpha \in R$, is called the **Weyl group** corresponding to the root system R. It is a finite subgroup of $O(n, \mathbb{R})$.

Note that conditions R1 and R2 are invariant under similarities (i.e. rotations and dilations) of $\mathbb{R}^n$. We shall not distinguish between two root systems that are similar to one another.

3a. If this condition is not satisfied, we can simply replace $\mathbb{R}^n$ by a smaller space $\mathbb{R}^m$ spanned by R. The number m is called the **rank** of the root system R.

3b. If $R_1 \sqcup R_2$ and $R_1 \perp R_2$, then both subsets R_1 and R_2 are themselves root systems and we can study them separately.

3c. We propose the following

Exercise 5. a) Let $\alpha \in R$ and $k\alpha \in R$ for some $k \in \mathbb{R}$. Then $k = \pm 1, \pm 2$, or $\pm\frac{1}{2}$.

b) Show that all possibilities above are indeed realized. (Hint: consider root systems of rank 1.) ♣

It is worthwhile to mention that for any $n \in \mathbb{Z}_+$ there is only one indecomposable non-reduced root system of rank n. It is denoted by $\mathbf{BC}_n$ and consists of vectors $\pm e_i \pm e_j$, $1 \le i \ne j \le n$, and $\pm e_k$ and $\pm 2e_k$, $1 \le k \le n$.

3d. The simply-laced root systems are also called ADE-systems (because of their classification described below). They appear in a larger number of classification problems than general root systems.

Now we introduce some basic definitions for general root systems.[10]

The complement to the union of all mirrors in $\mathbb{R}^n$ splits into several connected components $\overset{\circ}{C}_i$, called **open Weyl chambers**; their closures C_i are called simply **Weyl chambers**.

We say that a vector $\lambda \in \mathbb{R}^n$ is **regular** if it lies in an open Weyl chamber and **singular** if it belongs to at least one mirror.

A **linear order** in $\mathbb{R}^n$ is an order relation that is compatible with the structure of a real vector space, i.e. such that the sum of positive vectors is positive as well as a positive multiple of a positive vector.

Exercise 6. Show that any linear order relation is the lexicographical one with respect to an appropriate (not necessarily orthogonal) basis in $\mathbb{R}^n$.

Hint. Show that for any linear order in $\mathbb{R}^n$ there exists a hyperplane such that its complement splits into open half-spaces, one of which consists of positive vectors and the other of negative ones. Then apply induction. ♣

Every linear order in $\mathbb{R}^n$ induces an order relation in the root system R. We denote by R_+ (resp. R_-) the set of positive (resp. negative) roots. Clearly, there are finitely many order relations on R. To formulate a more precise statement, recall that for any cone $V \subset \mathbb{R}^n$ one can define the **dual cone** V' as the set of vectors v' satisfying the inequality $(v', v) \ge 0$ for all $v \in V$.

Proposition 3. *For any linear order in $\mathbb{R}^n$ the convex cone generated by R_+ is exactly the dual cone to one of the Weyl chambers.* □

Later on we shall fix an order relation and denote by C_+ the **positive Weyl chamber** defined by

$$C_+ = \{\lambda \in \mathbb{R}^n \mid (\lambda, \alpha) \ge 0 \text{ for all } \alpha \in R_+\}. \tag{28}$$

We call a root $\alpha \in R_+$ **decomposable** if it can be written in the form $\alpha = \beta + \gamma$ where both summands are also from R_+. Otherwise, we call α a **simple root**.

[10] Some of these definitions we already used in Appendix II.2.2 for a special root system in connection with finite-dimensional representations of $GL(n, \mathbb{R})$.

Lemma 5. *Any root $\alpha \in R$ has a unique decomposition $\alpha = \sum_{k=1}^{n} c_k \alpha_k$ where all α_k are simple roots. Moreover, the coefficients c_k are either all from $\mathbb{Z}_+$ or all from $\mathbb{Z}_-$.*

Proof. It is clear from the very definition of simple roots that any positive root is a linear combination of simple roots with non-negative integer coefficients. Hence, any negative root is a linear combination of simple roots with non-positive coefficients.

It remains to prove the uniqueness, i.e. linear independence of simple roots. We derive it from the following geometric property: two simple roots α, β never form an acute angle (or, in algebraic form: the scalar product (α, β) is never positive).

Assume the contrary. Then the number $k = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$ is a positive integer and if we assume that $(\alpha, \alpha) \geq (\beta, \beta)$, this number can only be 1. Thus, $s_\alpha \beta = \beta - \alpha$. If $s_\alpha \beta \in R_+$, then $\beta = s_\alpha \beta + \alpha$ is not simple; if $s_\alpha \beta \in R_-$, then $\alpha = \beta - s_\alpha \beta$ is not simple. Contradiction.

We return to the proof of the lemma. Suppose there is a non-trivial linear relation between simple roots: $\sum_k c_k \alpha_k = 0$. Put the terms with positive coefficients c_i on one side and the terms with negative coefficients c_j on the other. We get a relation of the form

$$v := \sum_{i \in I} a_i \alpha_i = \sum_{j \in J} b_j \alpha_j =: w$$

with positive coefficients a_i, b_j.

Since the relation is non-trivial, we have $v = w \neq 0$, hence $|v|^2 = |w|^2 > 0$. On the other hand,

$$(v, w) = \sum_{i \in I, j \in J} a_i b_j (\alpha_i, \alpha_j) \leq 0,$$

which is impossible. □

We denote by Π the set $\{\alpha_1, \ldots, \alpha_n\}$ of simple roots. Observe that C_+ is bounded by mirrors $M_1, \ldots, M_n$, corresponding to simple roots.

The whole root system R can be reconstructed from the system Π of simple roots. The proof of this and many other facts (including Proposition 3) is based on the following fundamental fact.

Proposition 4. *The group W acts simply transitively on the set of Weyl chambers.*

The proof is based on the following lemma, which is of independent interest.

Lemma 6. *Let $\lambda \in \overset{\circ}{C}_+$, and let $\mu \in \mathbb{R}^n$ be an arbitrary vector. Let $W(\mu)$ denote the W-orbit of μ. Then $W(\mu)$ has a unique common point with C_+, which is the nearest point to λ of $W(\mu)$.*

Sketch of the proof. Assume for simplicity that μ itself is the nearest point to λ among all $\{w \cdot \mu,\ w \in W\}$. If $\mu \notin C_+$, it is separated from λ by a mirror M_β, $\beta \in \Pi$. But in this case the point $s_\beta\mu$ is strictly closer to λ than μ. Conversely, if $\mu \in C_+$, one can show that for any $w \in W$ the difference $\mu - w \cdot \mu$ is a linear combination of simple roots with non-negative coefficients. It follows that μ is the only nearest point to λ. □

Corollary 1. *The group W acts simply transitively on the set of all linear order relations on R. So, the number of order relations on R and the number of Weyl chambers are both equal to the order of the Weyl group.*

Corollary 2. *The stabilizer in W of any vector $\lambda \in \mathbb{R}^n$ is generated by reflections in the mirrors that contain λ. In particular, the stabilizer of a regular vector $\lambda \in \mathbb{R}^n$ is the trivial subgroup $\{e\}$.*

To describe the system $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of simple roots, it is convenient to introduce the so-called **Cartan matrix** $A \in \mathrm{Mat}_n(\mathbb{Z})$ with entries

$$A_{i,j} = \frac{2(\alpha_i,\, \alpha_j)}{(\alpha_i,\, \alpha_i)}, \quad 1 \le i, j \le n. \tag{29}$$

The famous result by Dynkin claims that all the information about a given root system R is contained in its Cartan matrix A.

Since W acts on $\mathbb{R}^n$ by orthogonal transformations, Proposition 4 implies that the Cartan matrix does not depend on the choice of the order relation (hence on the choice of Π).

There exists a simple graphic way to present the information encoded in a Cartan matrix A. Introduce the **Dynkin graph** Γ_A with vertices labelled by simple roots, or just by numbers $1, 2, \ldots, n$. The two different vertices i and j are joined by $n_{i,j} = A_{i,j} \cdot A_{j,i}$ edges. If $|\alpha_i| > |\alpha_j|$, we add an arrow directed from i to j.

The same graph without arrows is sometimes called a **Dynkin diagram**.

The following properties of a Cartan matrix A show that it can be reconstructed from the graph Γ_A.

Proposition 5. a) *The diagonal elements of A are all equal to 2.*

b) *The off-diagonal elements of A are non-positive and satisfy the condition*

$$A_{i,j} = 0 \iff A_{j,i} = 0.$$

c) *All principal minors of A are positive. In particular, the quantity $n_{i,j} = A_{i,j} \cdot A_{j,i}$ can take only four values:* 0, 1, 2, *or* 3. $\square$

Thus, we have a chain of objects that define each other up to natural isomorphism:

$$R(\text{root system}) \leftrightarrow \Pi(\text{set of simple roots}) \leftrightarrow A(\text{Cartan matrix}) \\ \leftrightarrow \Gamma_A(\text{Dynkin graph})$$

Example 4. Here we list all Cartan matrices of size 2×2 and the corresponding graphs, and we also draw the reduced root systems of rank 2:

$$\begin{array}{ccccccc} A: & \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} & \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} & \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} & \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} & \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \\ \Gamma_A: & \circ \quad \circ & \circ\!-\!\!\!-\!\circ & \circ \Longrightarrow \circ & \circ \Longleftarrow \circ & \circ \Rrightarrow \circ & \circ \Lleftarrow \circ \end{array}$$

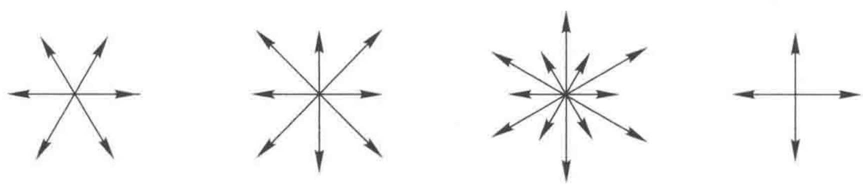

Figure 3. Reduced root systems of rank 2.

$\diamondsuit$

Example 5. The system $\mathbf{A_n}$. Let $\{e_i\}_{0 \le i \le n}$ be the standard basis in $\mathbb{R}^{n+1}$. Put

$$R = \{\alpha_{ij} := e_i - e_j \mid 0 \le i \ne j \le n\},$$

the set with $2 \cdot \binom{n}{2} = n^2 - n$ elements. This root system is degenerate of rank n. We leave it to the reader to check that it is indecomposable, reduced, and simply laced.

Let $x^0, \dots, x^n$ be the standard coordinates in $\mathbb{R}^{n+1}$. The mirrors M_{ij} are given by the equation $x^i = x^j$. They split $\mathbb{R}^{n+1}$ into $(n+1)!$ Weyl chambers. If we choose the standard lexicographical order in $\mathbb{R}^{n+1}$, then $R_+ = \{\alpha_{ij} \mid i < j\}$ and the positive Weyl chamber C_+ is defined by inequalities $x^0 \ge x^1 \ge \dots \ge x^n$.

The Weyl group is generated by permutations $x^i \leftrightarrow x^j$ and coincides with the symmetric group S_{n+1}.

Lemma 7. *The system of simple roots for* $\mathbf{A_n}$ *is* $\Pi = \{\alpha_k := \alpha_{k-1,k} \mid 1 \le k \le n\}$.

Proof. If $j > i+1$, then $\alpha_{ij} = \alpha_{i,i+1} + \alpha_{i+1,j}$. So, a positive root α_{ij} can be simple only if $j = i+1$. On the other hand, the number of simple roots is at least n, the rank of the system. Hence, all roots α_k are simple. □ ♢

Example 6. The system $\mathbf{D_n}$. Let $R \subset \mathbb{R}^n$, $n \ge 2$, be the set $\{\pm e_i \pm e_j \mid i \ne j\}$. It contains $4 \cdot \binom{n}{2} = 2n^2 - 2n$ vectors. This is a non-degenerate root system of rank n. It is also indecomposable, reduced, and simply-laced. The mirrors for this system are given by the equations $x^i = \pm x^j$. The Weyl group consists of all permutations of coordinates and all changes of an even number of their signs. As an abstract group, it is a semidirect product $S_n \ltimes \mathbb{Z}_2^{n-1}$.

With respect to the standard ordering, the positive Weyl chamber is given by inequalities $x^1 \ge \cdots \ge x^{n-1} \ge |x^n|$.

Exercise 7. Show that the system of simple roots for $\mathbf{D_n}$ is

$$\Pi = \{e_k - e_{k+1} \mid 1 \le k \le n-1\} \cup \{e_{n-1} + e_n\}.$$

Hint. Follow the proof of Lemma 6 and use the relation

$$(e_i + e_j) = (e_i - e_{j+1}) + (e_j + e_{j+1}) \quad \text{for } i < j < n.$$

♣ ♢

Example 7. The system $\mathbf{E_8}$. This is the most complicated simple root system. It has many realizations from which we choose only two:

1. Let $R \subset \mathbb{R}^8$ be the set of $4 \cdot \binom{8}{2} + 2^7 = 112+128 = 240$ vectors:

$$\{\pm e_i \pm e_j \mid i \ne j\} \cup \left\{ \sum_{i=1}^{8} \epsilon_i e_i \;\middle|\; \epsilon_i = \pm 1 \,\&\, \prod_{i=1}^{8} \epsilon_i = 1 \right\}.$$

2. Consider $2 \cdot \binom{9}{2} = 72$ vectors of the form $e_i - e_j$, $0 \le i \ne j \le 8$, and $2 \cdot \binom{9}{3} = 168$ vectors of the form $\pm(e_i + e_j + e_k)$ with different i, j, k. The projections of these 240 vectors on the hyperplane $\sum_{k=0}^{k=9} x^k = 0$ form a root system, isomorphic to system 1.

These two realizations show that $\mathbf{E_8}$ contains $\mathbf{D_8}$ and $\mathbf{A_8}$ as subsystems.

Exercise 8.* Show that $\mathbf{E_8}$ is a reduced indecomposable simply-laced root system of rank 8. Find the simple roots.

Answer: In the first realization:

$\alpha_k = e_{k+1} - e_{k+2}$, $1 \le k \le 6$, $\alpha_7 = e_7 + e_8$, $\alpha_8 = \frac{1}{2}(e_1 - e_2 - \cdots - e_7 + e_8)$.

In the second realization:

$\alpha_k = e_k - e_{k+1}$, $1 \le k \le 7$, $\alpha_8 = \frac{1}{3}(e_0 + e_1 + \cdots + e_8) - e_1 - e_2 - e_3$.

♣ ♢

All root systems are classified. To formulate the result, we observe that if the Dynkin graph Γ of R is not connected, then the system R is a direct sum of its orthogonal subsystems corresponding to the connected components of Γ. So, it is enough to classify all connected Dynkin graphs corresponding to indecomposable root systems. Also, keeping in mind the application to Lie groups, we restrict ourselves by reduced systems.

Theorem 6. *The connected Dynkin graphs corresponding to reduced root systems form four infinite series and five isolated examples. They are drawn in Figure 4.* □

$$\mathbf{A_n}: \quad \circ - \circ - \cdots - \circ, \quad n \geq 1, \qquad \mathbf{B_n}: \quad \circ - \circ - \cdots - \circ - \circ \Longrightarrow \circ, \quad n \geq 2,$$

$$\mathbf{C_n}: \quad \circ - \circ - \cdots - \circ \Longleftarrow \circ, \quad n \geq 3, \qquad \mathbf{D_n}: \quad \circ - \circ - \cdots - \overset{\overset{\circ}{|}}{\circ} - \circ, \quad n \geq 4,$$

$$\mathbf{E_n}: \quad \circ - \circ - \cdots - \circ - \overset{\overset{\circ}{|}}{\circ} - \circ - \circ, \quad n = 6,\ 7,\ 8.$$

$$\mathbf{F_4}: \quad \circ - \circ \Longrightarrow \circ - \circ, \qquad \mathbf{G_2}: \quad \circ \Rrightarrow \circ.$$

Figure 4. Dynkin graphs of indecomposable reduced root systems.

Exercise 9. Compute the determinants of the Cartan matrices for the series $\mathbf{A_n}$, $\mathbf{B_n}$, $\mathbf{C_n}$, $\mathbf{D_n}$, $\mathbf{E_n}$.

Hint. Show that $\det_n$ satisfies the recurrent relation

$$\det\nolimits_{n+1} = 2\det\nolimits_n - \det\nolimits_{n-1},$$

hence is a linear function of n. Also use the isomorphisms $\mathbf{A_1} \cong \mathbf{B_1} \cong \mathbf{C_1}$, $\mathbf{B_2} \cong \mathbf{C_2}$, $\mathbf{A_3} \cong \mathbf{D_3}$, $\mathbf{E_5} \cong \mathbf{D_5}$, $\mathbf{E_4} = \mathbf{A_4}$, $\mathbf{E_3} \cong \mathbf{A_2} + \mathbf{A_1}$.

Answer: In the self-explanatory notation we have $a_n = n + 1$, $b_n = c_n = 2$, $d_n = 4$, $e_n = 9 - n$. ♣

Now we discuss some properties of root systems related to the Weyl group. Define the **length** of $w \in W$ by the formula:

$$l(w) = \#\left(w(R_+) \cap R_-\right). \tag{30}$$

In words: $l(w)$ is the number of positive roots α that w transforms to a negative root.

Proposition 6. *The number $l(w)$ is equal to the minimal length of the decomposition*

$$w = s_{i_1} s_{i_2} \cdots s_{i_k}$$

into the product of canonical generators. The decomposition of minimal length is called **reduced**.

Proof. Let λ be any interior point of the positive Weyl chamber C_+. Consider a path joining λ with $w\lambda$. This path intersects some mirrors. Choose the path with a minimal number of intersections. Suppose this number is m and the path goes through the open Weyl chambers $\overset{\circ}{C}_+ = \overset{\circ}{C}_0, \overset{\circ}{C}_1, \ldots, \overset{\circ}{C}_m$. Choose representatives $\lambda_i \in \overset{\circ}{C}_i$ so that λ_{i-1} and λ_i are symmetric with respect to the mirror separating $\overset{\circ}{C}_{i-1}$ and $\overset{\circ}{C}_i$. Then all points $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_m = w(\lambda)$ have the form $\lambda_j = w_j(\lambda)$.

Moreover, since $w_j\lambda$ and $w_{j-1}\lambda$ belong to adjacent Weyl chambers, the same is true for λ and $w_j^{-1}w_{j-1}\lambda$. But we already observed that C_+ is bounded by mirrors corresponding to simple roots. Therefore $w_j^{-1}w_{j-1} = s_{i_j}$ and $w_j = w_{j-1}s_{i_j}$. We get the decomposition $w_j = s_{i_1}s_{i_2} \cdot s_{i_j}$.

Thus, to any path joining λ with $w \cdot \lambda$ and intersecting m mirrors there corresponds the reduced decomposition of w into a product of m generators. We proved that the length of a reduced decomposition of $w \in W$ is equal to the number of mirrors separating λ and $w \cdot \lambda$ for any regular vector λ.

Now we use the following

Lemma 8. *For $\lambda \in \overset{\circ}{C}_+$ the inequality $(\alpha, \lambda) > 0$ is equivalent to $\alpha \in R_+$.*

Proof of the lemma. By definition, $\overset{\circ}{C}_+$ consists of points λ satisfying $(\alpha, \lambda) > 0$ for all $\alpha \in R_+$.

Conversely, if $(\alpha, \lambda) > 0$, then $\alpha \in R_+$, since otherwise $\alpha \in R_-$ and (α, λ) is negative. □

Suppose now that a mirror M_α separates λ and $w \cdot \lambda$. Replacing, if necessary, the root α by $-\alpha$, we can assume that $(\alpha, \lambda) < 0$ but $(w^{-1} \cdot \alpha, \lambda) = (\alpha, w \cdot \lambda) > 0$. From Lemma 8 we conclude that $\alpha \in R_-$, $w^{-1} \cdot \alpha \in R_+$. Hence, w sends the positive root $w^{-1} \cdot \alpha$ to the negative root α.

Conversely, if w sends a positive root α to a negative one, then the mirror $M_{w \cdot \alpha}$ separates λ and $w \cdot \lambda$. Therefore, the number of mirrors, separating λ and $w \cdot \lambda$, is exactly the length of w. □

Corollary. *The generators s_i have length 1, i.e. only one positive root α has the property $s_i\alpha \in R_-$. (Certainly, this is the simple root α_i.)*

To each root system of rank n (or rather to the corresponding Weyl group W) one can associate the series of integers $e_1 \leq e_2 \leq \cdots \leq e_n$ that are called **exponents** and have many remarkable properties.[11] We mention some of them here.

Proposition 7. a) *The distribution of elements of W according to their length is given by the following generating functions:*

$$\sum_{w\in W} t^{l(w)} = \prod_{i=1}^{n} \frac{1-t^{e_i+1}}{1-t} = \prod_{i=1}^{n} (1+t+\cdots+t^{e_i}). \tag{31}$$

b) *The algebra of W-invariant polynomials on $\mathbb{R}^n$ is freely generated by homogeneous polynomials $P_1, \ldots, P_n$ such that* deg $P_i = e_i + 1$.

c) *The algebra of W-invariant elements in the Grassmann algebra $\bigwedge(\mathbb{R}^n)$ is freely generated by homogeneous elements $Q_1, \ldots, Q_n$ such that* deg $Q_i = 2e_i + 1$.

d) *The exponents satisfy the following relations:*

1) $\prod_{i=1}^{n} (e_i + 1) = |W|$; 2) $\sum_{i=1}^{n} e_i = |R_+|$; 3) $e_i + e_{n-i+1} = h$

where h is the order of the so-called **Coxeter element** $c = s_1 s_2 \cdots s_n \in W$.[12] □

To a root system R there correspond two important **lattices** (i.e. discrete subgroups) in $\mathbb{R}^n$: the so-called **root lattice** freely generated by simple roots:

$$Q = \mathbb{Z}\cdot\alpha_1 + \cdots + \mathbb{Z}\cdot\alpha_n \tag{32}$$

and the **weight lattice**

$$P = \left\{\lambda \in \mathbb{R}^n \mid \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \text{ for all } \alpha \in \Pi\right\}. \tag{33}$$

The weight lattice P is freely generated by **fundamental weights** $\omega_1, \ldots, \omega_n$ defined by

$$\frac{2(\omega_j, \alpha_i)}{(\alpha_i, \alpha_i)} = \delta_{ij}. \tag{34}$$

In terms of the Cartan matrix the relation between simple roots and fundamental weights can be written in the form

$$\alpha_j = \sum_i \omega_i A_{i,j} \qquad \text{or, symbolically,} \qquad \alpha = \omega\cdot A. \tag{35}$$

In particular, (39) implies that Q is a sublattice in P. Moreover, it has a finite index in P, i.e. the quotient group P/Q is finite and $\#(P/Q) = \det A$.

[11] There is a more general notion of **generalized exponents** introduced by B. Kostant. See also [**Ki13**] and references therein.

[12] In fact, this product depends on the choice of Π and on the numeration of simple roots, but all these elements belong to the same conjugacy class in W, hence have the same order.

3.2. Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.

This Lie algebra enters as a building block in all other complex semisimple Lie algebras. Therefore, it is important to know in detail the structure of this Lie algebra and of the corresponding Lie group $SL(2, \mathbb{C})$, as well as their representation theory. We recall here the basic facts of this theory.

Proposition 8. a) *Any finite-dimensional representation of* $\mathfrak{sl}(2, \mathbb{C})$ *is a direct sum of irreducible representations.*

b) *For any integer* $n \geq 0$ *there is exactly one equivalence class* π_n *of irreducible representations with* $\dim \pi_n = n + 1$.

In particular, π_0 *is the trivial representation* $\pi_0(X) = 0$, π_1 *is the defining (tautological) representation* $\pi_1(X) = X$, *and* π_2 *is the adjoint representation* $\pi_2(X) = \operatorname{ad} X$.

c) *The representation* π_n *is equivalent to the* n*-th symmetric power of* π_1.

d) *There are the following isomorphisms:*

$$\pi_m \otimes \pi_n = \bigoplus_{s=0}^{\min(m,n)} \pi_{m+n-2s}; \quad S^2(\pi_n) = \bigoplus_{s=0}^{n} \pi_{2n-2s};$$

$$\bigwedge^2(\pi_n) = \bigoplus_{s=1}^{n} \pi_{2n-2s+1}.$$

Proof. We start with the explicit construction of irreducible representations. Observe that the group $G = SL(2, \mathbb{C})$ acts from the right on the space $\mathbb{C}^2$ of row vectors

$$(x, y) \mapsto (x, y) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (\alpha x + \gamma y, \beta x + \delta y).$$

Let V be the space of polynomials in the variables x, y. Then we have a representation π of the group G in V:

$$\big(\pi(g)P\big)(x, y) = P(\alpha x + \gamma y, \beta x + \delta y) \quad \text{for } g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G.$$

Let us compute the corresponding action of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Assume that $X \in \mathfrak{g}$ has the form $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. Then $\exp \tau X = \begin{pmatrix} 1 + \tau a & \tau b \\ \tau c & 1 - \tau a \end{pmatrix} + o(\tau)$ and we get

$$\pi_*(X) = \frac{\partial}{\partial \tau} \pi(\exp \tau X) \Big|_{\tau=0} = a(x\partial_x - y\partial_y) + by\partial_x + cx\partial_y.$$

The representations (π, V) of G and (π_*, V) of $\mathfrak{g}$ are infinite-dimensional but they split into the sum of finite-dimensional representations since V is the direct sum of invariant subspaces V_n consisting of homogeneous polynomials of degree n.

It is clear that $(\pi_n, V_n) = S^n(\pi_1, V_1)$ because $V_n = S^n(V_1)$. We show that all these representations are irreducible.

Let $\{E, F, H\}$ be the canonical basis of $\mathfrak{sl}(2, \mathbb{C})$ (see Example 8 in Section 3.4 below). Then the formula above for π_* implies

$$\pi_n(E) = x\partial_y, \qquad \pi_n(F) = y\partial_x, \qquad \pi_n(H) = x\partial_x - y\partial_y.$$

Choose in V_n the natural monomial basis

$$v_k = x^{n-k}y^k, \quad 0 \le k \le n.$$

The representation π_n is given in this basis by the formulae:

$$\text{(36)} \quad \pi_n(E)v_k = kv_{k-1}, \quad \pi_n(F)v_k = (n-k)v_{k+1}, \quad \pi_n(H)v_k = (n-2k)v_k.$$

Exercise 10. Prove that all π_n are irreducible.

Hint. Show that starting from any non-zero polynomial $P \in V_n$ and acting several times by $\pi_n(E)$ we obtain the monomial cx^n with $c \neq 0$. Then, acting by $\pi_n(F)$, we get all other monomials. ♣

Thus, we have constructed a family $\{\pi_n\}$, $0 \le n < \infty$, of irreducible representations of $\mathfrak{g}$. They are pairwise non-equivalent, since they have different dimensions. The next step is to prove that any finite-dimensional irreducible representation of $\mathfrak{g}$ is equivalent to some of the representations π_n.

Let (ρ, W) be any finite-dimensional representation of $\mathfrak{g}$. Consider an eigenvector $w \in W$ for the operator $\rho(H)$, and let $\lambda \in \mathbb{C}$ be the corresponding eigenvalue. From the commutation relations (43) we obtain:

$$\begin{aligned}\rho(H)\rho(E)w &= \rho(E)\rho(H)w + [\rho(H), \rho(E)]w \\ &= \lambda\rho(E)w + 2\rho(E)w = (\lambda+2)\rho(E)w.\end{aligned}$$

In other words, the vector $\rho(E)w$, if non-zero, is also an eigenvector for $\rho(H)$ with the eigenvalue $\lambda+2$. It follows that vectors w, $\rho(E)w$, $\rho(E)^2w$, ... are linearly independent until they are zero because they are eigenvectors for $\rho(H)$ with different eigenvalues. Since $\dim V < \infty$, we have $\rho(E)^k w = 0$ for some $k \ge 1$. Choose the minimal k for which this is true and denote $\rho(E)^{k-1}w$ by w_0'. It is an eigenvector for $\rho(H)$ and we denote the corresponding eigenvalue $\lambda + 2k - 2$ by λ_0.

Now, introduce the vectors $w'_k := \rho(F)^k w'_0$. The same argument shows that all w'_k are eigenvectors for $\rho(H)$ with eigenvalues $(\lambda_0 - 2k)$. Hence, vectors w'_0, w'_1, ... are independent until they are zero. Let w'_n be the last non-zero vector in this sequence.

Exercise 11. Prove that the linear span W' of $w'_0, \ldots, w'_n$ is a $\mathfrak{g}$-invariant subspace in W.

Hint. The invariance of W' with respect to $\rho(F)$ and $\rho(H)$ follows from the very construction. Show by induction on k and using (43) that

$$\rho(E)w'_k = c_k w'_{k-1} \qquad \text{with } c_k = k(\lambda_0 - k + 1).$$

♣

If (ρ, W) is irreducible, we conclude that W' coincides with W. The spectrum of $\rho(H)$ consists of eigenvalues $(\lambda_0 - 2k)$, $0 \le k \le n$. On the other hand, $\operatorname{tr}\rho(H) = \operatorname{tr}[\rho(E), \rho(F)] = 0$. It follows that $\lambda_0 = n = \dim V - 1$. We come to the following final formula:

$$\rho(E)w'_k = k(n-k+1)w'_{k-1}, \qquad \rho(F)w'_k = w'_{k+1}, \qquad \rho(H)w'_k = (n-2k)w'_k.$$

Comparing this with (36), we see that the map $w'_k \mapsto \frac{1}{(n-k)!}v_k$ establishes the equivalence between (ρ, W) and (π_n, V_n).

It remains to show that any finite-dimensional representation of $\mathfrak{g}$ is a direct sum of irreducible representations. There are two ways to do this.

The first, algebraic, way uses the quadratic Casimir element Δ in the universal enveloping algebra $U(\mathfrak{g})$. We note that a dual basis to E, F, H in $\mathfrak{g}$ with respect to the Killing form is $\frac{1}{4}F$, $\frac{1}{4}E$, $\frac{1}{8}H$. Hence, $\Delta = \frac{1}{8}(H^2 + 2EF + 2FE)$.

An easy computation shows that $\pi_n(\Delta) = \frac{n(n+2)}{8} \cdot 1_{V_n}$. Therefore, the spectral decomposition of $\pi_n(\Delta)$ in V gives us the decomposition of V into isotypic components for $\mathfrak{g}$.

The second, analytic, way uses the equivalence of representation categories for $\mathfrak{g}$, for its compact real form $\mathfrak{su}(2, \mathbb{C})$ and for the corresponding compact group $SU(2, \mathbb{C})$ (see Chapter 5). The latter category is evidently semisimple: any object is a direct sum of simple (indecomposable) objects. □

3.3. Root system related to $(\mathfrak{g}, \mathfrak{h})$.

We start with the formulation of basic facts and main structure theorems for semisimple Lie algebras.

For any Lie algebra $\mathfrak{g}$ the following bilinear form $(\cdot\,, \cdot)_K$ was first defined by W. Killing and extensively used by Elie Cartan; we call it the **Killing form**:

$$(X, Y)_K := \operatorname{tr}(\operatorname{ad} X \cdot \operatorname{ad} Y). \tag{37}$$

It is clearly invariant under all automorphisms of $\mathfrak{g}$.

For solvable Lie algebras this form is practically of little use because it vanishes on any nilpotent ideal and, in particular, on the derived Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. On the contrary, for semisimple Lie algebras it is a very important tool.

Theorem 7 (E. Cartan). *The Killing form on a Lie algebra* $\mathfrak{g}$ *is non-degenerate iff* $\mathfrak{g}$ *is semisimple.*

Indeed, if $\mathfrak{g}$ has a non-zero abelian ideal $\mathfrak{a}$, then it is in the kernel of the Killing form. Conversely, if $\mathfrak{g}$ has no abelian ideals, then it is a direct sum of non-abelian simple ideals, hence is semisimple.

So, it remains to show that a simple Lie algebra has non-degenerate Killing form. It follows from the fact (which we do not prove here) that the kernel of the Killing form is a nilpotent ideal in $\mathfrak{g}$. □

For any semisimple Lie algebra $\mathfrak{g}$ we fix the G-invariant form $(\cdot\ ,\ \cdot)$ on $\mathfrak{g}^*$, which is dual to the Killing form on $\mathfrak{g}$.

The main structure theorem for semisimple Lie algebras is the following.

Proposition 9. *Let* $\mathfrak{g}$ *be a complex semisimple Lie algebra.*

a) *There exists a* **canonical decomposition** *of* $\mathfrak{g}$ *into a direct sum of three vector subspaces, which are subalgebras, but not ideals in* $\mathfrak{g}$*:*

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \tag{38}$$

such that for any linear finite-dimensional representation (π, V) *of* $\mathfrak{g}$*, after suitable choice of a basis in* V*, the elements of these subspaces go respectively to lower triangular, diagonal, and upper triangular matrices in* $\mathfrak{sl}(m, \mathbb{C})$*,* $m = \dim V$*.*

b) *The decomposition* (38) *is unique up to an inner automorphism of* $\mathfrak{g}$*.*

c) *The subalgebra* $\mathfrak{h}$ *is called a* **Cartan subalgebra** *and is characterized by the property: it is a maximal abelian subalgebra consisting of ad-semisimple elements.*[13] □

Applying Proposition 9 to the adjoint representation $(\mathrm{ad}, \mathfrak{g})$ of $\mathfrak{g}$, we see that all operators $\mathrm{ad}\, H$, $H \in \mathfrak{h}$, are simultaneously diagonalizable. It follows that

$$\mathfrak{g} = \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha$$

where A is a finite subset in $\mathfrak{h}^*$ and $\mathfrak{g}_\alpha$ consists of elements $X \in \mathfrak{g}$ satisfying

$$[H, X] = \alpha(H) \cdot X \quad \text{for all } H \in \mathfrak{h}. \tag{39}$$

[13] An element X is called ad-semisimple if the operator $\mathrm{ad}\, X$ in an appropriate basis is written as a diagonal matrix.

It is clear that $\mathfrak{g}_0 = \mathfrak{h}$ (since $\mathfrak{h}$ is a maximal abelian subalgebra).

Exercise 12. Show that

a) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \begin{cases} \subseteq \mathfrak{g}_{\alpha+\beta} & \text{for } \alpha+\beta \in A, \\ = 0 & \text{for } \alpha+\beta \notin A. \end{cases}$

b) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta)_K = 0$ if $\alpha + \beta \neq 0$.

Hint. Use the relation (39) and the Jacobi identity. ♣

It follows that the Killing form establishes the duality between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ and, in particular, is non-degenerate on $\mathfrak{h} = \mathfrak{g}_0$.

Let us denote by $R(\mathfrak{g}, \mathfrak{h})$ or simply by R the set of non-zero elements in A. Denote by V the real vector subspace in $\mathfrak{h}^*$ spanned by R.

The relation between complex semisimple Lie groups on one hand and abstract root systems on the other hand can be formulated as follows

Theorem 8. a) *The bilinear form $(\cdot, \cdot)$ is positive definite on V and $R(\mathfrak{g}, \mathfrak{h})$ is a reduced non-degenerate root system in the Euclidean space V.*

b) *Any abstract reduced non-degenerate root system is obtained from a certain pair $(\mathfrak{g}, \mathfrak{h})$.*

c) *Two complex semisimple Lie algebras are isomorphic iff they have isomorphic root systems (i.e. isomorphic Dynkin diagrams).*

Sketch of the proof. The proof is heavily based on the representation theory of the simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ (see the previous section).

For any root α we choose a non-zero vector $X_\alpha \in \mathfrak{g}_\alpha$. From the invariance of the Killing form we conclude that

$$([X_\alpha, X_{-\alpha}], H)_K = (X_\alpha, [X_{-\alpha}, H])_K = \alpha(H)(X_\alpha, X_{-\alpha})_K.$$

Since the Killing form is non-degenerate, we can assume that $(X_\alpha, X_{-\alpha})_K = 1$. Then the element $\check{\alpha} := [X_\alpha, X_{-\alpha}]$ has the property $\check{\alpha}, H)_K = \alpha(H)$ for all $H \in \mathfrak{h}$. In other words, $\check{\alpha}$ is exactly the element of $\mathfrak{h}$ that corresponds to $\alpha \in \mathfrak{h}^*$ under the isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$ induced by the Killing form.

The element $\check{\alpha}$ is sometimes called the **dual root** to α. The collection of all dual roots forms a root system which is denoted by $R^\vee$ and is called the **dual root system**.

Let $\mathfrak{g}(\alpha)$ denote the 3-dimensional subalgebra in $\mathfrak{g}$ spanned by X_α, $X_{-\alpha}$, and $\check{\alpha}$. It is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ with the canonical basis $E = X_\alpha$, $H = H_\alpha$, $F = X_{-\alpha}$ where $H_\alpha := \frac{2\check{\alpha}}{(\alpha, \alpha)}$.

Let us study the adjoint action of $\mathfrak{g}(\alpha)$ in $\mathfrak{g}$. Choose a root β and consider the so-called **α-string** of β, i.e. the set of roots of the form $\{\beta + k\alpha, \ -p \leq$

$k \leq q\}$ such that the root vectors $X_k \in \mathfrak{g}_{\beta+k\alpha}$ satisfy the relations

$$[X_\alpha, X_k] = \begin{cases} c_k X_{k+1} & \text{for } k < q, \\ 0 & \text{for } k = q, \end{cases}$$
$$[X_{-\alpha}, X_k] = \begin{cases} c'_k X_{k-1} & \text{for } k > -p, \\ 0 & \text{for } k = -p, \end{cases}$$

with some non-zero constants $c_k,\ c'_k$.

It is clear that the subspace $S(\beta) \subset \mathfrak{g}$ spanned by elements $\{X_k\}_{-p\leq k\leq q}$ is an irreducible $\mathfrak{g}(\alpha)$-module. From the results of the previous section it follows that for some $n \in \mathbb{Z}_+$ we have $q + p = n$ and

$$[H_\alpha, X_{\beta+k\alpha}] = (2k + p - q)X_{\beta+k\alpha}.$$

Since $H_\alpha = \frac{2\check{\alpha}}{(\alpha,\alpha)}$, we conclude that

$$2(\alpha, \beta) = (p - q)(\alpha, \alpha) \quad \text{or} \quad \frac{2(\alpha, \beta)}{(\alpha, \alpha)} = p - q \in \mathbb{Z}.$$

So, we have verified the first axiom of root systems and have also given a representation-theoretic interpretation of the integers $A_{\alpha,\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$.

To check the second axiom, we consider again the Lie subalgebra $\mathfrak{g}(\alpha)$ and define the element $\widetilde{s}_\alpha \in \exp \mathfrak{g}(\alpha) \subset G$ by

$$\widetilde{s}_\alpha = \exp \frac{\pi}{2}(X_\alpha - X_{-\alpha}).$$

The corresponding inner automorphism of $\mathfrak{g}$ preserves all subspaces of type $S(\beta)$ and interchanges the extreme vectors X_{-p} and X_q. Since the roots $\beta - p\alpha$ and $\beta + q\alpha$ are symmetric with respect to the mirror M_α, we conclude that the corresponding automorphism of $\mathfrak{h}^*$ is just the reflection s_α.

As a byproduct we get the realization of the abstract Weyl group as the group $W = N_G(\mathfrak{h})/H$ of automorphisms of $\mathfrak{h}$ and of $\mathfrak{h}^*$. In Chapter 5 we saw that W is also the group of automorphisms of full flag manifolds.

The proofs of statements b) and c) are more involved and we omit them. We refer the reader to [**Bou**], [**FH**], [**GG**], [**OV**]. □

The decomposition (42) defines the additional structure on $R(\mathfrak{g}, \mathfrak{h})$: the splitting of R into two disjoint subsets R_+ and $R_- = -R_+$ defined by the relations

$$\mathfrak{n}_+ = \bigoplus_{\alpha\in R_+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_- = \bigoplus_{\alpha\in R_-} \mathfrak{g}_\alpha.$$

The roots from R_+ (resp. R_-) are called **positive roots** (resp. **negative roots**).

We denote by r the dimension of $\mathfrak{n}_\pm$ (i.e. the cardinality of $R_\pm$) and by $n =: \operatorname{rk}\mathfrak{g}$ the dimension of $\mathfrak{h}$. These numbers are related by the equality $2r + n = \dim\mathfrak{g}$.

Let $\alpha_1, \ldots, \alpha_n$ be the simple roots. Let $H_i = \alpha_i^\vee$, $1 \le i \le n$, be the dual roots. They are characterized by the property

$$(40) \qquad \alpha_j(H_i) = A_{i,j} \quad \text{or} \quad \omega_j(H_i) = \delta_{ij} \quad \text{for all } j = 1, \ldots, n.$$

The second important result is

Proposition 10. a) *For any $\alpha \in R$ the subspace $\mathfrak{g}_\alpha$ is 1-dimensional, hence is spanned by a single element X_α, which is called a* **root vector**.

b) *The root vectors X_α can be normalized so that the following relations hold:*

$$(41) \qquad [X_\alpha, X_\beta] = \begin{cases} N_{\alpha,\beta}\, X_{\alpha+\beta} & \text{when } 0 \neq \alpha + \beta \in R, \\ 0 & \text{when } 0 \neq \alpha + \beta \notin R, \\ H_\alpha & \text{when } \alpha + \beta = 0 \end{cases}$$

where $N_{\alpha,\beta}$ are non-zero integers satisfying $N_{-\alpha,-\beta} = -N_{\alpha,\beta} = N_{\beta,\alpha}$. □

Corollary. a) *The real span of elements X_α, $X_{-\alpha}$, H_α for all $\alpha \in R$ is a real form of $\mathfrak{g}$. It is called the* **normal** *or* **split** *real form and is denoted by $\mathfrak{g}_n$.*

b) *The real span of elements $\frac{X_\alpha - X_{-\alpha}}{2}$, $\frac{iH_\alpha}{2}$, $\frac{X_\alpha + X_{-\alpha}}{2i}$ for all $\alpha \in R$ is also a real form of $\mathfrak{g}$. It is called the* **compact** *real form and is denoted by $\mathfrak{g}_c$.*

Informally speaking, this corollary tells us that every complex semisimple Lie algebra $\mathfrak{g}$ is in a sense "built" from r copies of $\mathfrak{sl}(2, \mathbb{C})$, while its normal (resp. compact) real form is constructed from r copies of $\mathfrak{sl}(2, \mathbb{R})$ (resp. $\mathfrak{su}(2, \mathbb{C})$).

Therefore, a good understanding of these three Lie algebras is very essential for the whole theory. In particular, the simplest proofs of Propositions 6–9 can be obtained by considering the restrictions of different representations (π, V) of $\mathfrak{g}$ to subalgebras $\mathfrak{g}(\alpha)$, $\alpha \in R$.

Now we are in a position to formulate the classification theorem for simple Lie algebras. As usual, we start with the complex case.

Theorem 9. *Every complex semisimple Lie algebra $\mathfrak{g}$ is a direct sum of simple Lie algebras. There are four infinite series of complex simple Lie*

algebras, which are called **classical** *simple Lie algebras:*

$\mathbf{A_n} \cong \mathfrak{sl}(n+1,\ \mathbb{C}),\ n \geq 1;$ $\qquad$ $\mathbf{B_n} \cong \mathfrak{so}(2n+1,\ \mathbb{C}),\ n \geq 2,$

$\mathbf{C_n} \cong \mathfrak{sp}(2n,\ \mathbb{C}),\ n \geq 3,$ $\qquad$ $\mathbf{D_n} \cong \mathfrak{so}(2n,\ \mathbb{C}),\ n \geq 4,$

and five isolated examples: $\mathbf{G_2}$, $\mathbf{F_4}$, $\mathbf{E_n}$, $n = 6, 7, 8$, *which are called* **exceptional** *simple Lie algebras. The corresponding root systems are described in Chapter* 5. □

The index n here denotes the rank of $\mathfrak{g}$ which was defined above as the dimension of a Cartan subalgebra $\mathfrak{h}$. It is also equal to the codimension of a generic adjoint orbit.[14]

All classical simple Lie algebras have explicit matrix realizations:

$\mathfrak{sl}(n+1,\ \mathbb{C})$ — all traceless complex matrices of order $n+1$;

$\mathfrak{so}(n,\ \mathbb{C})$ — all complex antisymmetric matrices X of order n.

$\mathfrak{sp}(2n,\ \mathbb{C})$ — all complex matrices X of order $2n$ which satisfy the equation $X^t J_{2n} + J_{2n} X = 0$ (or the equivalent condition: $S := J_{2n} X$ is symmetric).

Remark 2. The classical Lie algebras $\mathbf{A_n}, \mathbf{B_n}, \mathbf{C_n}, \mathbf{D_n}$ are defined for all natural numbers n. The restrictions on n in Theorem 9 are made to avoid the appearance of non-simple or isomorphic Lie algebras. ♡

Exercise 13. Prove the following isomorphisms:

a) $\mathbf{B_1} \simeq \mathbf{C_1} \simeq \mathbf{A_1}$; b) $\mathbf{C_2} \simeq \mathbf{B_2}$; c) $\mathbf{D_1} \simeq \mathbb{C}^1$; d) $\mathbf{D_2} \simeq \mathbf{A_1} \oplus \mathbf{A_1}$; e) $\mathbf{D_3} \simeq \mathbf{A_3}$.

Hint. Compare the corresponding root systems. See also Chapter 5. ♣

We will not give the description of exceptional Lie algebras here. The interested reader can get some information about them in Chapter 5.

Here we discuss only some basic notions which are needed to understand the main ideas. We hope that this will allow our readers to not only get the general impression about this theory, but to also use the basic results in their research. For proofs and further information we refer to the books [**Bou**], [**FH**], [**Hu**], and [**OV**].

3.4. Real forms.

For a real Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ we define its **complexification** $(\mathfrak{g}_\mathbb{C}, [\cdot,\cdot]_\mathbb{C})$ as a complex vector space $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ with bilinear operation $[\cdot,\ \cdot]_\mathbb{C}$ extended from $[\cdot,\ \cdot]$ by complex linearity. Simply speaking, passing from $\mathfrak{g}$ to $\mathfrak{g}_\mathbb{C}$ means that we keep the same structure constants but allow complex linear combinations of basic vectors.

[14] If we replace "adjoint" by "coadjoint", we get the definition of rank for an arbitrary Lie algebra. See Chapter 1 for details.

Now let $\mathfrak{g}$ be a complex Lie algebra. We say that a real Lie algebra $\mathfrak{g}_0$ is a **real form** of $\mathfrak{g}$ if $\mathfrak{g}$ is isomorphic to $(\mathfrak{g}_0)_{\mathbb{C}}$ as a complex Lie algebra.

Remark 3. A given complex Lie algebra $\mathfrak{g}$ can have no real form at all or have several different real forms (non-isomorphic as real Lie algebras).

Let $O \subset \mathcal{A}_n(\mathbb{C})$ be the $GL(n, \mathbb{C})$-orbit corresponding to $\mathfrak{g}$ (see the previous section). The existence of a real form means that O has non-empty intersection with $\mathcal{A}_n(\mathbb{R})$. In other words, for an appropriate choice of basis all structure constants are real.

The general theory of real algebraic groups says that $O \cap \mathcal{A}_n(\mathbb{R})$ splits into a finite number of $GL(n, \mathbb{R})$-orbits. It means that for any complex Lie algebra $\mathfrak{g}$ there exist only finitely many real forms (up to isomorphism). ♡

Example 8. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ be the 3-dimensional complex Lie algebra of traceless 2×2 matrices. It has two remarkable bases with real structure constants:

1st canonical basis: $X = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $Y = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $Z = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

with commutation relations

$$[X, Y] = Z, \qquad [Y, Z] = X, \qquad [Z, X] = Y, \tag{42}$$

and

2nd canonical basis: $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

with commutation relations

$$[E, F] = H, \qquad [H, E] = 2E, \qquad [H, F] = -2F. \tag{43}$$

Exercise 14. Check that the two bases above define different real Lie algebras.

Hint. Consider the signature of the **Killing** bilinear form in $\mathfrak{g}$:

$$(X, Y) := \operatorname{tr}(\operatorname{ad} X \cdot \operatorname{ad} Y). \tag{44}$$

♣

We see that $\mathfrak{g}$ has at least two different real forms: one isomorphic to $\mathfrak{su}(2)$ and the other isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Actually, $\mathfrak{sl}(2, \mathbb{C})$ has no other real forms.

The description of real simple Lie algebras is based on the following fact.

Theorem 10. a) *Every complex simple Lie algebra remains simple when considered as a real Lie algebra.*

b) *Every complex simple Lie algebra has a finite number* (≥ 2) *of real forms that are simple real Lie algebras.*

c) *Every real simple Lie algebra is obtained from a complex simple Lie algebra by the procedures described in* a) *or* b). □

We use the following standard notation to list all the real forms of classical simple complex Lie algebras:

$\mathfrak{sl}(n, \mathbb{R})$ – the set of all traceless real matrices of order n;

$\mathfrak{sl}(n, \mathbb{H})$ – the set of all quaternionic matrices X of order n that satisfy the condition $\operatorname{Re}\operatorname{tr} X = 0$;

$\mathfrak{sp}(2n, \mathbb{R})$ – the set of all real matrices X of order $2n$ that satisfy the equation $X^t J_{2n} + J_{2n} X = 0$ (or the equivalent condition: $S := J_{2n} X$ is symmetric);

$\mathfrak{so}^*(2n)$ (denoted also as $\mathfrak{u}^*(n)$) – the set of all quaternionic matrices X of order n that satisfy the equation $\alpha(X)^* J_{2n} + J_{2n}\alpha(X) = 0$ (the equivalent condition: $S := J_{2n}\alpha(X)$ is Hermitian); here α denotes the embedding of $\operatorname{Mat}_n(\mathbb{H})$ into $\operatorname{Mat}_{2n}(\mathbb{C})$;

$\mathfrak{so}(p, q, \mathbb{R})$ – the set of all real matrices X of order $n = p + q$ that satisfy the equation $X^t I_{p,q} + I_{p,q} X = 0$;

$\mathfrak{su}(p, q, \mathbb{C})$ – the set of all complex traceless matrices X of order $n = p + q$ that satisfy the equation $X^* I_{p,q} + I_{p,q} X = 0$;

$\mathfrak{su}(p, q, \mathbb{H})$ (denoted also as $\mathfrak{sp}(p, q)$) – the set of all quaternionic matrices X of order $n = p+q$ that satisfy the equations $X^* I_{p,q} + I_{p,q} X = 0$, $\operatorname{Re}\operatorname{tr} X = 0$.

In the last three cases when $p = n$, $q = 0$ the shorter notation $\mathfrak{so}(n)$, $\mathfrak{su}(n)$, $\mathfrak{sp}(n)$ is used.

The list of real forms.

$\mathbf{A_n}$: $\mathfrak{sl}(n + 1, \mathbb{R})$; $\mathfrak{su}(p, q, \mathbb{C})$, $p \leq q$, $p + q = n + 1$; $\mathfrak{sl}(\frac{n+1}{2}, \mathbb{H})$ for n odd.

$\mathbf{B_n}$: $\mathfrak{so}(p, q, \mathbb{R})$, $p \leq q$, $p + q = 2n + 1$.

$\mathbf{C_n}$: $\mathfrak{sp}(2n, \mathbb{R})$; $\mathfrak{sp}(p, q) \simeq \mathfrak{su}(p, q, \mathbb{H})$, $p \leq q$, $p + q = n$.

$\mathbf{D_n}$: $\mathfrak{so}(p, q, \mathbb{R})$, $p \leq q$, $p + q = 2n$; $\mathfrak{so}^*(n)$ for n even.

Recall that among all the real forms of a given complex simple Lie algebra $\mathfrak{g}$ there are two of special interest (cf. Corollary in Section 3.3).

The compact form $\mathfrak{g}_c$ is characterized by the following equivalent properties:

a) it admits a matrix realization as a subalgebra of $\mathfrak{su}(m)$;

b) the quadratic form $Q(X) = \mathrm{tr}\,(\mathrm{ad}\,X)^2$ is negative definite.

The normal form $\mathfrak{g}_n$ is also characterized by two equivalent properties:

a) it admits a real canonical decomposition, i.e. a real matrix realization such that

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \tag{38'}$$

where the three subspaces are intersections of $\mathfrak{g}$ with lower triangular, diagonal, and upper triangular subalgebras;

b) the quadratic form $Q(X) = \mathrm{tr}\,(\mathrm{ad}\,X)^2$ has the signature $(\frac{N+n}{2},\ \frac{N-n}{2})$ where $N = \dim\,\mathfrak{g}$, $n = \mathrm{rk}\,\mathfrak{g} := \dim\mathfrak{h}$.

Exercise 15. Locate compact and normal forms in the above list of real forms.

Hint. Compute the signature of the quadratic form $Q(X) = \mathrm{tr}\,(\mathrm{ad}\,X)^2$. ♣

Exercise 16. Establish the following isomorphisms of the real forms corresponding to the isomorphisms from Exercise 13.

a) $\mathfrak{su}(2,\,\mathbb{C}) \simeq \mathfrak{so}(3,\,\mathbb{R}) \simeq \mathfrak{su}(1,\,\mathbb{H})$, $\mathfrak{sl}(2,\,\mathbb{R}) \simeq \mathfrak{so}(2,\,1,\,\mathbb{R}) \simeq \mathfrak{sp}(2,\,\mathbb{R})$.

b) $\mathfrak{sp}(4,\,\mathbb{R}) \simeq \mathfrak{so}(3,\,2,\,\mathbb{R})$, $\mathfrak{su}(2,\,\mathbb{H}) \simeq \mathfrak{so}(5,\,\mathbb{R})$, $\mathfrak{su}(1,\,1,\,\mathbb{H}) \simeq \mathfrak{so}(4,\,1,\,\mathbb{R})$.

c) $\mathfrak{so}(2,\,\mathbb{R}) \simeq \mathfrak{so}(1,\,1,\,\mathbb{R}) \simeq \mathbb{R}^1$.

d) $\mathfrak{so}(4,\,\mathbb{R}) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, $\mathfrak{so}(3,\,1,\,\mathbb{R}) \simeq \mathfrak{sl}(2,\,\mathbb{C})$, $\mathfrak{so}(2,\,2,\,\mathbb{R}) \simeq \mathfrak{su}(1,\,1,\,\mathbb{C})$, $\mathfrak{so}^*(4) \simeq \mathfrak{so}(3,\,\mathbb{R}) \oplus \mathfrak{sl}(2,\,\mathbb{R})$.

e) $\mathfrak{so}(6,\,\mathbb{R}) \simeq \mathfrak{su}(4,\,\mathbb{C})$, $\mathfrak{so}(5,\,1,\,\mathbb{R}) \simeq \mathfrak{sl}(2,\,\mathbb{H})$, $\mathfrak{so}(4,\,2,\,\mathbb{R}) \simeq \mathfrak{su}(2,\,2,\,\mathbb{C})$, $\mathfrak{so}(3,\,3,\,\mathbb{R}) \simeq \mathfrak{sl}(4,\,\mathbb{R})$, $\mathfrak{so}^*(6) \simeq \mathfrak{su}(3,\,1,\,\mathbb{C})$.

There is one more isomorphism, obtained from the extra symmetry of the Dynkin graph $\mathbf{D_4}$: $\mathfrak{so}^*(8) \simeq \mathfrak{so}(6,\,2,\,\mathbb{R})$. ♣

4. Homogeneous manifolds

4.1. G-sets.

Here we recall some known facts mainly to establish convenient terminology to be used later.

Let G be a group. The set X is called a **left G-set** if a map

$$G \times X \to X : \qquad (g,\ x) \mapsto g \cdot x$$

is given which satisfies the condition

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x, \qquad e \cdot x = x.$$

This means that we have a group homomorphism $G \to \operatorname{Aut} X$ that associates to $g \in G$ the transformation $x \mapsto g \cdot x$. In this case we say also that G **acts on X from the left**. The homomorphism $G \to \operatorname{Aut} X$ is called **faithful** if it has the trivial kernel: no element except e acts as identity. We call such an action **effective**.

Sometimes the notion of a **right G-set** or **right action** is used. It is defined as a map

$$X \times G \to X : \qquad (x, g) \mapsto x \cdot g$$

that satisfies the condition

$$(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2), \qquad x \cdot e = x.$$

This means that the map $G \to \operatorname{Aut} X$ that sends $g \in G$ to the transformation $x \mapsto x \cdot g$ is an **antihomomorphism** (i.e. reverses the order of factors).

Often the set X in question has an additional structure preserved by the group action; e.g., X can be a vector space or a smooth manifold. Then we shall call it a **G-space** or **G-manifold**, etc.

The collection of all left (or right) G-sets forms a category $\mathcal{G}\text{-}\mathcal{S}ets$. Objects of this category are G-sets and morphisms are so-called **G-equivariant maps** $\phi : X \to Y$ for which the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "g"'] & Y \arrow[d, "g"] \\ X \arrow[r, "\phi"] & Y. \end{tikzcd}$$

Example 9. The usual matrix multiplication defines a left action of $GL(n, \mathbb{R})$ on the space $\mathbb{R}^n$ of column vectors and a right action of the same group on the space $(\mathbb{R}^n)^*$ of row vectors:

$$v \mapsto Av, \quad f \mapsto fA, \qquad v \in \mathbb{R}^n,\ f \in (\mathbb{R}^n)^*,\ A \in GL(n, \mathbb{R}).$$

$\diamondsuit$

Example 10. For any category K the set $\operatorname{Mor}_K(A, B)$ is a left G-space for $G = \operatorname{Aut} A$ and a right G-space for $G = \operatorname{Aut} B$. $\diamondsuit$

Remark 4. In fact it is easy to switch from left action to right action (and vice versa) using the antiautomorphism $g \mapsto g^{-1}$ of the group G. Namely, on any left G-space X we can canonically define a right action by the rule

$$x \cdot g := g^{-1} \cdot x.$$

It follows that the categories of left and right G-spaces are equivalent.

Later on we often omit "right" and "left" and speak just on G-sets, keeping in mind that each of them can be considered in both ways as explained above. In any formula it will be clear from the notation which kind of action we have in mind. ♡

We introduce (or recall) some more definitions and notation.

(i) There is a natural operation of a **direct product** in the category of G-sets: it is the usual direct product of sets endowed by the diagonal action of G:

$$g \cdot (x, y) = (g \cdot x, g \cdot y).$$

(ii) For any G-set X we shall denote by X/G or X_G the set of G-**orbits** in X.

(iii) We denote by X^G the set of **fixed points** for G, i.e. those points $x \in X$ for which $g \cdot x = x$ for all $g \in G$.

(iv) For any two G-sets X and Y their **fibered product over** G is defined as $(X \times Y)/G$ and denoted by $X \underset{G}{\times} Y$.

Here it is convenient to assume that X is a right G-set and Y is a left G-set. Then the set $X \underset{G}{\times} Y$ is a quotient of $X \times Y$ with respect to the equivalence relation $(x \cdot g, y) \sim (x, g \cdot y)$. If we denote the class of (x, y) by $x \underset{G}{\times} y$, then the equivalence above takes the form of an associativity law:

$$(x \cdot g) \underset{G}{\times} y = x \underset{G}{\times} (g \cdot y). \tag{45}$$

Note that in general there is no group action on $X \underset{G}{\times} Y$. But in the case when X or Y is endowed by an action of another group H that commutes with a G-action there is a natural H-action on the product space $X \underset{G}{\times} Y$. In particular, this is the case when G is abelian and any G-set can also be considered as a $(G \times G)$-set.

Let H be a subgroup of G. Then the categories $\mathcal{G}$-$\mathcal{S}ets$ and $\mathcal{H}$-$\mathcal{S}ets$ are related by two functors:

The **restriction functor** $\mathrm{res}_H^G : \mathcal{G}\text{-}\mathcal{S}ets \to \mathcal{H}\text{-}\mathcal{S}ets$ is defined in the obvious way: any G-set is automatically an H-set.

The dual **induction functor** $\mathrm{ind}_H^G : \mathcal{H}\text{-}\mathcal{S}ets \to \mathcal{G}\text{-}\mathcal{S}ets$ is defined on the objects as

$$\mathrm{ind}_H^G X = G \underset{H}{\times} X. \tag{46}$$

Here G is considered as a left G-set and a right H-set. So the product is a left G-set. If $\psi : Y_1 \to Y_2$ is an H-equivariant map, we define $\mathrm{ind}_H^G \psi$ by the

formula

$$\operatorname{ind}_H^G\psi(g \underset{H}{\times} y) = g \underset{H}{\times} \psi(y). \tag{47}$$

We highly recommend that the reader verify that ind_H^G is indeed a functor. The most important part of this assignment is to formulate in "down-to-earth" terms what you have to verify.

The two functors res_H^G and ind_H^G are related by

Theorem 11 (Reciprocity Principle). *For any $X \in Ob(\mathcal{G}\text{-}\mathcal{S}ets)$ and any $Y \in Ob(\mathcal{H}\text{-}\mathcal{S}ets)$ there is a natural bijection:*

$$\operatorname{Mor}_{\mathcal{G}\text{-}\mathcal{S}ets}(\operatorname{ind}_H^G Y,\, X) \cong \operatorname{Mor}_{\mathcal{H}\text{-}\mathcal{S}ets}(Y,\, \operatorname{res}_H^G X). \tag{48}$$

Proof. Let $\phi \in \operatorname{Mor}_{\mathcal{H}\text{-}\mathcal{S}ets}(Y,\, \operatorname{res}_H^G X)$. This means that ϕ is a map from Y to X that is H-equivariant: $\phi(h \cdot y) = h \cdot \phi(y)$.

On the other hand, an element $\Phi \in \operatorname{Mor}_{\mathcal{G}\text{-}\mathcal{S}ets}(\operatorname{ind}_H^G Y,\, X)$ is a G-equivariant map from $G \underset{H}{\times} Y$ to X, i.e. $\Phi(g_1 g \underset{H}{\times} y) = g_1 \cdot \Phi(g \underset{H}{\times} y)$. We leave it to the reader to check that the formulae

$$\phi(y) := \Phi(e \underset{H}{\times} y), \qquad \Phi(g \underset{H}{\times} y) := g \cdot \phi(y) \tag{49}$$

establish the required bijection $\phi \leftrightsquigarrow \Phi$. □

Remark 5. Sometimes ind_H^G is called a **left adjoint functor** to res_H^G because the equality (48) looks like the well-known formula for the adjoint operator in a space with an inner product:

$$(A^* x,\, y) = (x,\, Ay).$$

In our case the role of a (set-valued) inner product in a category $\mathcal{C}$ is played by the bifunctor

$$\mathcal{C} \times \mathcal{C} \to \mathcal{S}et: \quad X, Y \mapsto \operatorname{Mor}_{\mathcal{C}}(X,\, Y).$$

♡

Example 11. a) If X is the group H acting on itself by right shifts, then $\operatorname{ind}_H^G X$ is the group G acting on itself by left shifts.

b) If X is a one-point set with trivial action of H, then $\operatorname{ind}_H^G X$ is the set G/H of left H-cosets in G (see below). ♢

This example shows the geometric meaning of the induction functor in some special cases. We shall come back to it in the next section.

A G-set X is called **homogeneous** if for any two points x_1, x_2 of X there is an element $g \in G$ that sends x_1 to x_2. This is equivalent to the statement: X/G is a one-point set. In this case we also say that G acts **transitively** on X.

The following simple facts are frequently used.

Lemma 9. *There are natural one-to-one correspondences*

a) *between homogeneous G-sets with a marked point and subgroups of G;*

b) *between homogeneous G-sets and conjugacy classes of subgroups of G.*

Proof. a) Let X be a homogeneous G-set with a marked point $x_0 \in X$. We can associate to this data the subgroup $H = Stab(x_0) \subset G$, which is called the **stabilizer** of x_0 and consists of those $g \in G$ that fix the point x_0.

Conversely, to any subgroup $H \subset G$ we can define the set $X = G/H$ of left H-cosets in G with the marked point $x_0 = H$.

(Recall that **left H-cosets** are precisely H-orbits in G if we consider the group G as a right H-space; so, the general H-coset is $xH = \{xh \mid h \in H\}$.)

We leave it to the reader to verify that the correspondences $(X, x_0) \rightsquigarrow Stab(x_0)$ and $H \rightsquigarrow (G/H, H)$ are reciprocal.

b) Let X be a homogeneous G-space with no marked point. We observe that all subgroups $Stab(x)$, $x \in X$, belong to the same conjugacy class, which we denote by C. This follows from the relation $Stab(g \cdot x) = g\, Stab(x)\, g^{-1}$.

Conversely, if a conjugacy class C of subgroups in G is given, we can choose a representative $H \in C$ and define the homogeneous G-set $X = G/H$. It remains to check two things:

1) that different choices of $H \in C$ produce isomorphic homogeneous G-sets;

2) the correspondences $X \rightsquigarrow C$ and $C \rightsquigarrow X$ constructed above are reciprocal.

We leave the checking to the reader. □

Let $X = G/H$ be a homogeneous G-set. We want to describe the group $\mathrm{Aut}(X)$ of all automorphisms of X (as an object of the category of G-sets).

Define the **normalizer** of H in G as $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$. It is clear that it is the maximal subgroup in G that has H as a normal subgroup.

Theorem 12. *The group* $\mathrm{Aut}(X)$ *is isomorphic to* $N_G(H)/H$.

Proof. Let $\phi \in \mathrm{Aut}(X)$. Then $\phi(H) = gH$ for some $g \in G$. Since ϕ is an automorphism of X, we get $\phi(g_1H) = g_1gH$ for any $g_1 \in G$. In particular, for $g_1 = h \in H$ we get $gH = \phi(hH) = hgH$. Therefore $HgH = gH$ and $g \in N_G(H)$. Conversely, for any $g \in N_G(H)$ the map $g_1H \mapsto g_1gH = g_1Hg$ is an automorphism of X.

It remains to observe that two elements g_1, g_2 of $N_G(H)$ define the same automorphism of X iff $g_2 \in g_1 H$. □

4.2. G-manifolds.

The main statements of the previous section remain true for Lie groups and homogeneous manifolds if we agree to consider only closed subgroups $H \subset G$.

In particular, it is true for Lemma 9 and Theorems 11 and 12. For example, we have

Lemma 10. *Let G be a Lie group. There is a natural one-to-one correspondence*

a) *between homogeneous G-manifolds with a marked point and closed subgroups $H \subset G$;*

b) *between homogeneous G-manifolds and conjugacy classes of closed subgroups $H \subset G$.*

The proof is a combination of Lemma 9 and Theorem 4 above. □

Example 12. Consider the 2-dimensional sphere $S^2 \subset \mathbb{R}^3$. It is a homogeneous space with respect to the group $G = SO(3, \mathbb{R})$. The stabilizer of the north pole is the subgroup $H = SO(2, \mathbb{R})$ naturally embedded in $SO(3, \mathbb{R})$ as a subgroup of matrices of the form

$$h(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The normalizer of H in G consists of two components: $N_0 = H$ and N_1, which is the set of matrices of the form

$$n(t) = \begin{pmatrix} -\cos t & \sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

So, $\mathrm{Aut}(S^2)$ consists of two points: the trivial automorphism, corresponding to N_0, and the antipodal map, corresponding to N_1. ♢

Example 13.* The previous example has a far-reaching generalization. Let K be a compact connected Lie group, and let $T \subset K$ be a maximal connected commutative subgroup of K usually called a **maximal torus**. It is well known that all such subgroups form a single conjugacy class. We define the **flag manifold** to be the homogeneous space $\mathcal{F} := K/T$.

This is one of the most beautiful examples of homogeneous manifolds. It is widely used in representation theory of semisimple groups, in topology, and in geometry.

The group $W = \operatorname{Aut} \mathcal{F}$ plays an important role. It is called the **Weyl group** associated to K. It turns out that $\mathcal{F}$ has a very rich geometric structure; e.g. it admits several K-invariant complex and almost complex structures. The group G of holomorphic transformations of $\mathcal{F}$ is a complex Lie group G in which K is a maximal compact subgroup. The stabilizer of a point $F \in \mathcal{F}$ in G is a maximal solvable subgroup $B \subset G$. The manifold $\mathcal{F}$ also admits a Kähler structure and is often used as a basic example in many geometric theories.

More details about this example are discussed in Chapter 5. $\diamondsuit$

Now we discuss the topology of homogeneous manifolds. Since a homogeneous manifold $M = G/H$ is determined by the pair (G, H), all topological problems about this manifold can in principle be reduced to pure algebraic questions.

Here we list some facts about homotopy groups and related homology and cohomology groups.

Let $X = G/H$ be a homogeneous manifold. Then we have the following exact sequence of homotopy groups (see Appendix I.2.3):

$$\begin{gathered} \cdots \to \pi_n(H) \to \pi_n(G) \to \pi_n(X) \to \pi_{n-1}(H) \to \cdots \\ \cdots \to \pi_1(H) \to \pi_1(G) \to \pi_1(X) \to \pi_0(H) \to \pi_0(G) \to \pi_0(X) \to \{1\}. \end{gathered} \tag{50}$$

Later on we shall usually assume that the group G in question is connected and simply connected. Then $\pi_0(G) = \pi_1(G) = \{1\}$. It is known that in this case we also have $\pi_2(G) = \{1\}$. So, (50) implies the following isomorphisms:

$$\pi_0(X) = \{1\}, \qquad \pi_k(X) \cong \pi_{k-1}(H), \qquad k = 1, 2. \tag{51}$$

We also recall that some homology and cohomology groups can easily be recovered from homotopy groups.

For example, for X connected the homology group $H_1(X) := H_1(X, \mathbb{Z})$ is just the **abelianization** of the homotopy group $\pi_1(X)$:

$$H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)]. \tag{52}$$

For X simply connected we also have

$$H_2(X) := H_2(X, \mathbb{Z}) \cong \pi_2(X). \tag{53}$$

If the homology groups $H_i(X)$ have no torsion (for instance, isomorphic to $\mathbb{Z}^k$ for some k), then the cohomology group with real coefficients has the form

$$H^i(X, \mathbb{R}) = \operatorname{Hom}(H_i(X), \mathbb{R}) \cong \mathbb{R}^{b_i(X)} \tag{54}$$

where $b_i(X)$ is the **i-th Betti number** of X.

For future use we formulate one corollary of these results as

Proposition 11. *Let G be a connected and simply connected Lie group acting on some smooth manifold M. A G-orbit $\Omega \subset M$ is simply connected iff the stabilizer G_F of a point $F \in \Omega$ is connected. If this is the case, we have a natural isomorphism:*

$$H^2(\Omega, \mathbb{R}) \cong H^1(G_F, \mathbb{R}) \cong \operatorname{Hom}(\pi_1(G_F), \mathbb{R}). \tag{55}$$

Sketch of the proof. We argue in terms of differential forms. A class $c \in H^2(\Omega, \mathbb{R})$ can be represented by a closed differential form σ. Let $p : G \to \Omega$ be the canonical projection: $p(g) = g \cdot F$. Then the form $p^*\sigma$ is closed, hence exact, because $H^2(G, \mathbb{R}) = \operatorname{Hom}(\pi_2(G), \mathbb{R}) = 0$.

Therefore $p^*\sigma = d\theta$ for some 1-form θ on G. The restriction on G_F gives us a 1-form $\theta_0 := \theta|_{G_F}$. This form is closed since $d\theta_0 = d\theta|_{G_F} = p^*\sigma|_{G_F} = p^*(\sigma|_{p(G_F)})$ and $p(G_F)$ is a point F. The class $[\theta_0]$ represents the image of c in $H^1(G_F, \mathbb{R}) \cong \operatorname{Hom}(\pi_1(G_F), \mathbb{R})$.

Conversely, let $\chi : \pi_1(H) \to \mathbb{R}$ be a homomorphism. Since $\pi_1(G_F) \cong \pi_2(\Omega)$ (see (55)), we get a homomorphism from $\pi_2(\Omega)$ to $\mathbb{R}$. But for a simply connected manifold Ω the homotopy group $\pi_2(\Omega)$ coincides with $H_2(\Omega)$. Thus we get a homomorphism from $H_2(\Omega)$ to $\mathbb{R}$, which is an element of $H^2(\Omega, \mathbb{R})$.

It remains to check that the two constructed maps are reciprocal. This is essentially the Stokes theorem: $\int_D d\theta = \int_{\partial D} \theta$ where D is a 2-dimensional film in G with ∂D a 1-cycle in G_F. □

Example 14. Let $G = SU(2)$, $H = U(1)$, and $X = P^1(\mathbb{C}) \cong S^2$. Since $SU(2) \cong S^3$ and $U(1) \cong S^1$ as smooth manifolds, we get from (55):

$$\pi_0(S^2) = \pi_1(S^2) = \{1\}, \qquad \pi_2(S^2) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

Proposition 11 implies in this case that $H^2(S^2, \mathbb{R}) \cong H^1(S^1, \mathbb{R}) \cong \mathbb{R}$.

4.3. Geometric objects on homogeneous manifolds.

Consider the following data.

(i) A smooth manifold M with the action of some group G on it; we denote the action by $m \mapsto g \cdot m$.

Usually G is a Lie group, but in some examples it is convenient to use an infinite-dimensional group in the role of G, such as $\operatorname{Diff}(M)$, the group of all diffeomorphisms of M, or a subgroup preserving some geometric structure on M.

(ii) A fiber bundle $F \to E \xrightarrow{p} M$, where F is a smooth manifold, possibly with some additional structure.

(iii) An action of the group G on the total space E, denoted by $x \to g \bullet x$ and compatible with the G-action on the base M. This means that for any $g \in G$ the following diagram is commutative:

$$\begin{array}{ccc} E & \xrightarrow{g\bullet} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ M & \xrightarrow[g\cdot]{} & M. \end{array}$$

When all three conditions (i), (ii), (iii) are satisfied, we shall say that the bundle $F \to E \xrightarrow{p} M$ is a G-**bundle**. If moreover the action of G on M is transitive, we say that $F \to E \xrightarrow{p} M$ is a **homogeneous** G-**bundle**.

We have seen in Appendix II.2.2 that geometric objects on a manifold M can be viewed as sections of some **natural bundle** on M, i.e. a bundle $F \to E \xrightarrow{p} M$ such that the action of the group $\mathrm{Diff}(M)$ can be lifted from M to E. So, a geometric object on M is just a section of a $\mathrm{Diff}(M)$-bundle.

Note that if M is connected, the group $\mathrm{Diff}(M)$ acts transitively on M: for any two points $m_1, m_2 \in M$ there exists a diffeomorphism $\phi \in \mathrm{Diff}(M)$ such that $\phi(m_1) = m_2$.[15] So, any $\mathrm{Diff}(M)$-bundle on a connected manifold is homogeneous.

Let m_0 be a marked point on M, let $F = p^{-1}(m_0)$ be the fiber over m_0, and denote by $H = \mathit{Stab}(m_0)$ the stabilizer of m_0 in G. The main fact about homogeneous G-bundles is that such a bundle is completely determined by the H-set F.

Indeed, consider the map

$$G \times F \to E: \ (g, x) \mapsto g \bullet x.$$

Since for any $x \in F$ we have $(gh) \bullet x = g \bullet (h \cdot x)$, this map can be factored through the fibered product $G \underset{H}{\times} F$ defined in the previous section. Moreover, the preimage in $G \times F$ of the point $g \bullet x$ is exactly the equivalence class $g \underset{H}{\times} x$. Thus, the total set E is identified with the fibered product $G \underset{H}{\times} F$.

Using the notation of Section 4.1, we can also write

$$E = \operatorname{ind}_H^G F. \tag{56}$$

[15]To see this, consider a smooth path joining m_1 and m_2 and a tiny tube along this path. The question for a general manifold reduces to the same question for a cylinder, which is rather easy.

This gives the geometric interpretation of the functor ind_H^G.

Denote by $\Gamma(E)$ the set of all smooth sections of E. For any G-bundle E we define the action of G on $\Gamma(E)$ by

$$(g \cdot s)(m) = g \bullet s(g^{-1} \cdot m). \tag{57}$$

In the case of a trivial bundle this is the standard action of the group G on the space of functions on the G-set $B : (g \cdot f)(m) = f(g^{-1} \cdot m)$.

A section $s : M \to E$ is called **G-invariant** (or simply **invariant** if there is no doubt about G) if $g \cdot s = s$, or $s(g \cdot m) = g \bullet s(m)$ for all $g \in G$.

Lemma 11. *Let $F \to E \xrightarrow{p} M$ be a homogeneous G-bundle, and let $H \subset G$ be the stabilizer of the point $m \in M$. There is a natural bijection between G-invariant sections of E and H-invariant elements in $F = p^{-1}(m)$.*

Proof. Indeed, let s be a G-invariant section. Then its value at m is an H-invariant element of F since $h \bullet s(m) = s(h \cdot m) = s(m)$. Conversely, if $x \in F$ is an H-invariant element, then the formula $s(g \cdot m) = g \bullet x$ defines correctly a section of E since the last expression depends only on the coset gH. This section is G-invariant because $(g_1 \cdot s)(g \cdot m) = g_1 \bullet s(g_1^{-1} \cdot g \cdot m) = g_1 g_1^{-1} g \cdot x = s(g \cdot m)$. □

Note that if G is a Lie group and the action of G on M is smooth and transitive, then any invariant section is automatically smooth.

Remark 6. Actually, the construction of a G-invariant section $s : M \to E$ from an H-invariant element $x \in F$ is nothing but the application of the functor ind_H^G to the morphism $\{m\} \to F : m \mapsto x$ in the category of H-sets. ♡

We give several useful applications of Lemma 11 here.

Example 15. We show the pure algebraic procedure to find out whether the connected homogeneous manifold $M = G/H$ is orientable. We can and will assume that the group G is connected.

Consider the fiber bundle $\{\pm 1\} \to E \xrightarrow{p} M$ for which the fiber F over any point $m \in M$ consists of two points ± 1 and the transition function $\phi_{\alpha,\beta}$ acts as

$$\phi_{\alpha,\beta}(\epsilon) = \operatorname{sign}(\det J_{\alpha,\beta}) \cdot \epsilon$$

where $J_{\alpha,\beta} = \left|\frac{\partial x_\alpha^i}{\partial x_\beta^j}\right|$ is the Jacobi matrix.

Further, we can define the G-action on E by

$$g \cdot (m,\, \epsilon) = (g \cdot m,\, \det g_*(m) \cdot \epsilon).$$

It should be clear for the reader (check if it is really so!) that the following statements are equivalent:

a) M is orientable;

b) the bundle E admits a continuous section $s: M \to E$;

c) the bundle E admits a G-invariant section $s: M \to E$;

d) E is trivial.

Indeed, a section s is continuous iff it is G-invariant. (Actually, both properties mean that s is locally constant in any trivialization of E.)

But according to Lemma 11, a G-invariant section exists iff there is an H-invariant element of the fiber, i.e. when $\det h_*(m) > 0$ for any $h \in H$. Note that the action of h_* on T_mM is simply the quotient action of $\operatorname{Ad} h$ on the quotient space $\mathfrak{g}/\mathfrak{h}$. So, we come to

Criterion 1. *Let G be a connected Lie group. Then the homogeneous manifold $M = G/H$ is orientable iff*

$$\frac{\det(\operatorname{Ad}_{\mathfrak{g}} h)}{\det(\operatorname{Ad}_{\mathfrak{h}} h)} > 0 \quad \textit{for all } h \in H. \tag{58}$$

In particular, this is certainly true if H is connected.

As an illustration, consider the case $M = P^n(\mathbb{R})$, $G = SO(n+1, \mathbb{R})$, $H = O(n, \mathbb{R})$. The element $h \in H$ has the form

$$h = \begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix}, \quad A \in O(n, \mathbb{R}).$$

The action of $h_*(m)$ on $T_mM \cong \mathfrak{g}/\mathfrak{h} \cong \mathbb{R}^n$ is given by

$$h_*(m) \cdot x = A \cdot x \cdot \det A.$$

Therefore, $\det h_*(m) = (\det A)^{n+1}$ and $P^n(\mathbb{R})$ is orientable iff n is odd.

Quiz. The statement above is false for $n = 0$. Where is the gap in our argument? ♢

Example 16. We want to find out when a G-invariant measure exists on a homogeneous space $M = G/H$. It turns out that this question can be solved in the same way as in the previous example. Here the auxiliary bundle E has the fiber $\mathbb{R}_{>0}$ and the transition function

$$\phi_{\alpha,\beta}(\rho) = \left| \det J_{\alpha,\beta} \right| \cdot \rho.$$

We leave the details to the reader and formulate the answer.

Criterion 2. *Let G be any Lie group. Then the homogeneous manifold $M = G/H$ admits a G-invariant measure iff*

$$\left| \frac{\det(\mathrm{Ad}_{\mathfrak{g}} h)}{\det(\mathrm{Ad}_{\mathfrak{h}} h)} \right| = 1 \quad \textit{for all } h \in H. \tag{59}$$

For example, this is true if H is either a compact, or a connected semisimple, or a nilpotent subgroup in G.

Note that if instead of the invariant measure, we were looking for the invariant differential form of top degree (the volume form) on M, then condition (59) will be replaced by the similar but stronger condition

$$\frac{\det(\mathrm{Ad}_{\mathfrak{g}} h)}{\det(\mathrm{Ad}_{\mathfrak{h}} h)} = 1 \quad \text{for all } h \in H. \tag{60}$$

As an illustration, we can cite the following example. The Möbius band M can be viewed as a quotient of $\mathbb{R}^2$ by the action of the group $\mathbb{Z}$. Namely, the element $k \in \mathbb{Z}$ sends $(x, y) \in \mathbb{R}^2$ to $\big(x + k, (-1)^k y\big)$. In this case condition (59) is satisfied, while (60) is not. Therefore, there exists an $\mathbb{R}^2$-invariant measure on M, but there is no $\mathbb{R}^2$-invariant 2-form. ♢

Example 17. Let us find out when the homogeneous manifold $M = G/H$ admits a G-invariant almost complex structure. Recall that an almost complex structure on M is just a complex structure on fibers of the tangent bundle TM. Analytically it is given by the smooth family of operators $J(m) \in \mathrm{End}(T_m M)$, $m \in M$ (in other words, by a smooth tensor field of type $(1,1)$ on M), satisfying $J^2 = -1$.

According to Lemma 11, the G-invariant almost complex structures on M are in bijection with the H-invariant operators J on $\mathfrak{g}/\mathfrak{h}$ satisfying $J^2 = -1$, hence with H-invariant complex structures on $T_{m_0}M$. ♢

The analogous question about G-invariant complex structures on $M = G/H$ is more delicate but also can be solved in pure algebraic terms. We recommend that the reader compare the result below with the discussion of Nijenhuis brackets in Appendix II.2.3.

Theorem 13. *Let G be a connected Lie group, and let H be a closed subgroup of G. Assume that the homogeneous manifold $M = G/H$ possesses a G-invariant almost complex structure J. Denote by P the i-eigenspace for J in the complexification $T_H^{\mathbb{C}} M \cong \mathfrak{g}^{\mathbb{C}}/\mathfrak{h}^{\mathbb{C}}$, and let $\mathfrak{p}$ be its preimage in $\mathfrak{g}^{\mathbb{C}}$. Then J is integrable iff the complex vector space $\mathfrak{p}$ is a Lie subalgebra in $\mathfrak{g}^{\mathbb{C}}$.* □

Not only G-invariant sections but all sections of a homogeneous G-bundle admit a simple analytic description. Namely, to a section $s : M \to E$ we

associate its **representing function** $f_s : G \to F$, given by

$$f_s(g) = g^{-1} \bullet s(g \cdot m_0) = (g^{-1} \cdot s)(m_0). \tag{61}$$

Lemma 12. a) *An F-valued function f on G is a representing function for a section of the homogeneous G-bundle $F \to E \xrightarrow{p} M$ iff it satisfies the* **covariance condition***:*

$$f(gh) = h^{-1} \bullet f(g). \tag{62}$$

b) *If the covariance condition is satisfied, the section s represented by f is given by*

$$s(g \cdot m) = g \bullet f(g). \tag{63}$$

c) *The G-action on sections in terms of representing functions has the form:*

$$f_{g \cdot s}(g_1) = f_s(g^{-1} g_1). \tag{64}$$

□

Remark 7. Equations (62) and (64) would look slightly better if we consider M as a right G-set but keep the left action on E. Namely, they take the form

$$f(hg) = h \bullet f(g), \tag{62$'$}$$

$$f_{g \cdot s}(g_1) = f_s(g_1 g). \tag{64$'$}$$

Of course, we have to change the definitions of a representing function and of the G-action on $\Gamma(E)$ in an appropriate way. Instead of (57) and (61) we would have

$$(g \cdot s)(m) = g \bullet s(m \cdot g), \tag{57$'$}$$

$$f_s(g) = s(m_0 \cdot g) \bullet g^{-1}. \tag{61$'$}$$

♡

Example 18. Vector fields on a Lie group. A Lie group G can be considered as a homogeneous G-space in two different ways: as a left G-space with respect to left shifts $x \mapsto gx$ and as a right G-space with respect to right shifts $x \mapsto xg$. Moreover, it can be considered as a $(G \times G)$-space with respect to the left action $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$.

Therefore, the sections of homogeneous vector bundles over G admit three different descriptions. We give these descriptions for the case of the tangent bundle TG.

A smooth section of TG, i.e. a vector field v on G, can be described as a smooth $\mathfrak{g}$-valued function on G in two different ways:

$$f_v^{left}(g) = g^{-1} \cdot v(g), \qquad f_v^{right}(g) = v(g) \cdot g^{-1}.$$

It is also described as a smooth $\mathfrak{g}$-valued function on $G \times G$:

$$F_v(g_1, g_2) = g_1^{-1} \cdot v(g_1 g_2^{-1}) \cdot g_2$$

satisfying the condition

$$F_v(g_1 g, g_2 g) = \operatorname{Ad} g^{-1} F_v(g_1, g_2).$$

These three descriptions are related by:

$$F_v(g_1, g_2) = \operatorname{Ad} g_1^{-1} f_v^{right}(g_1 g_2^{-1}) = \operatorname{Ad} g_2^{-1} f_v^{left}(g_1 g_2^{-1}).$$

The action of $G \times G$ on vector fields in terms of representing functions looks like:

$$f^{left}_{g_1 \cdot v \cdot g_2^{-1}}(g) = \operatorname{Ad} g_2\, f_v^{left}(g_1^{-1} g_2), \qquad f^{right}_{g_1 \cdot v \cdot g_2^{-1}}(g) = \operatorname{Ad} g_1\, f_v^{right}(g_1^{-1} g_2),$$
$$F_{g_1 \cdot v \cdot g_2^{-1}}(g_1', g_2') = F_v(g_1 g_1', g_2 g_2').$$

In particular, for the left-invariant vector field $\widetilde{X}$ we have

$$f_{\widetilde{X}}^{left}(g) \equiv X, \qquad f_{\widetilde{X}}^{right}(g) = \operatorname{Ad} g\, X, \qquad F_{\widetilde{X}}(g_1, g_2) = \operatorname{Ad} g_2^{-1} X.$$

For the right-invariant vector field $\widehat{X}$ we have:

$$f_{\widehat{X}}^{left}(g) = \operatorname{Ad} g^{-1} X, \qquad f_{\widehat{X}}^{right}(g) \equiv X, \qquad F_{\widehat{X}}(g_1, g_2) = \operatorname{Ad} g_1\, X.$$

$\diamondsuit$

In conclusion we collect some facts about integration on Lie groups.

On any Lie group G there exist unique (up to a scalar factor) left-invariant and right-invariant differential forms of top degree. We have denoted them $d_l g$ and $d_r g$ respectively. Put $\Delta(g) = \det \operatorname{Ad} g$.

Proposition 12. *The following relations hold:*

$$\begin{aligned} &d_l(hg) = d_l g; \qquad d_l(gh) = \Delta(h)^{-1} d_l g; \qquad d_l(g^{-1}) = c \cdot d_r g; \\ &d_r(hg) = \Delta(h) d_r g \qquad d_r(gh) = d_r g; \qquad d_r(g^{-1}) = c^{-1} \cdot d_l g. \end{aligned} \tag{65}$$

Proof. We prove only the first three relations; the proof of the others is quite analogous. The first relation is just the definition of the left-invariance.

To prove the second, we remark that the form $d_l(g)$ under the inner automorphism $A(h^{-1}) : g \mapsto h^{-1}gh$ goes to another left-invariant form, hence is simply multiplied by a scalar. The direct computation at the point e shows that this scalar is actually $\det \operatorname{Ad}(h)^{-1} = \Delta(h)^{-1}$. Therefore $d_l(gh) = d_l(h^{-1}gh) = \Delta(h)^{-1} d_l g$.

The third relation follows from the observation that the form $d_l(g^{-1})$ is right-invariant: $d_l((gh)^{-1}) = d_l(h^{-1}g^{-1}) = d_l(g^{-1})$. $\square$

Elements of Functional Analysis

1. Infinite-dimensional vector spaces

In this appendix we collect some definitions and results from functional analysis that are used in representation theory. Most of them have rather simple formulations and the reader acquainted with them can read the main text without difficulties. The proofs are often omitted, except in those cases when the proof is not difficult and/or helps to understand the matter and memorize the formulation.

1.1. Banach spaces.

In representation theory we often deal with infinite-dimensional vector spaces. The pure algebraic approach to such spaces is rather fruitless. A much more interesting and powerful theory arises when we combine algebraic methods with analysis and topology.

One of the main objects of this theory is the notion of Banach space, which is the combination of two structures: a real or complex vector space and a complete metric space. Here we recall some basic facts from the theory of Banach spaces and the linear operators on them.

A **Banach space** is a vector space V over the field $K = \mathbb{R}$ or $\mathbb{C}$ endowed with a distance function $d: V \times V \to \mathbb{R}_+$ such that

a) (V, d) is a complete metric space (see Appendix I.1.2);

b) the distance is compatible with the linear operations:

(i) $d(v + a, w + a) = d(v, w)$ (Translation invariance)

(ii) $d(cv, cw) = |c| \cdot d(v, w)$ for any $c \in K$ (Homogeneity)

These two distance properties allow us to replace this function of two variables by a function of one variable, the so-called **norm** of the vector v denoted by $\|v\|$ and defined by

$$\|v\| := d(v,\, 0).$$

The initial distance function can be reconstructed from the norm $d(x,\, y) = \|x - y\|$. The homogeneity of distance is equivalent to the homogeneity of norm:

$$\|cx\| = |c| \cdot \|x\|.$$

The triangle inequality for the distance is equivalent to the following basic property of the norm:

$$\|x + y\| \leq \|x\| + \|y\| \qquad \text{(Semiadditivity)}.$$

Example 1. The space $C(X,\, K)$ of continuous K-valued functions on a compact topological space X is a Banach space with respect to the norm

$$\|f\| = \max_{x \in X} |f(x)|. \tag{1}$$

Note the remarkable universal property of this space: any separable Banach space (i.e. possessing a countable dense subset) over K is isomorphic to a closed subspace in $C([0,\, 1],\, K)$. ♢

Example 2. The space $L^p(X,\, \mu,\, K)$ of measurable K-valued functions on a measure space $(X,\, \mu)$ is a Banach space with respect to the norm

$$\|f\|_p = \left(\int_X |f(x)|^p d\mu(x)\right)^{\frac{1}{p}},\ 1 \leq p \leq \infty. \tag{2}$$

For $p = \infty$ the norm is defined as

$$\|f\|_\infty = \lim_{p \to \infty} \|f\|_p = \operatorname{ess\,sup}_X |f(x)|. \tag{2'}$$

♢

Example 3. The space $C^k(M)$ of k-smooth K-valued functions on a smooth compact manifold M. In this case there is no preferable norm, but the corresponding class of convergent sequences is easy to describe.

A sequence $\{f_n\}$ converges to f in $C^k(M)$ if for any chart $U \subset M$ with a coordinate system $x^1,\, \ldots,\, x^m$ and for any multi-index $p = (p_1,\, \ldots,\, p_m)$ with $|p| \leq k$ the sequence $\partial^p f_n(x) = \partial_1^{p_1}, \ldots, \partial_m^{p_m} f_n(x)$ converges to $\partial^p f(x)$ uniformly on every compact subset of U.

We omit the explicit definition of a norm and the proof of completeness (see, e.g., [**KG**] or [**Y**]). $\diamondsuit$

Warning. The space $C^\infty(M)$, where the class of convergent sequences is defined in the same way, but using all multi-indices p, *is not a Banach space*. More precisely, there is no norm in $C^\infty(M)$ with the above given class of convergent sequences.

However, the topological space $C^\infty(M)$ is metrizable: one can define on $C^\infty(M)$ a metric such that it will be a complete metric space and the convergence defined by the metric will be exactly the convergence described above. This metric, however, is not homogeneous.

The space $C^\infty(M)$ for a non-compact manifold M is still more sophisticated: it is *non-metrizable* (see *loco cit.*). ♠

1.2. Operators in Banach spaces.

A linear operator A from one Banach space V_1 to another Banach space V_2 is called **bounded** if it has a finite **norm**, given by

$$\|A\| := \sup_{0 \neq x \in V_1} \frac{\|Ax\|_{V_2}}{\|x\|_{V_1}}. \tag{3}$$

It is well known that A is bounded iff it is continuous.

The collection of all bounded operators from V_1 to V_2 is denoted $\mathrm{Hom}\,(V_1, V_2)$. It is a Banach space with respect to the operator norm (3).

In particular, $V^* := \mathrm{Hom}(V, K)$ is a Banach space. It is called the **dual Banach space** to V and its elements are called **linear continuous functionals** (or, for short, **functionals**) on V.

Example 4. It is known that $L^p(X, \mu, K)^* \cong L^q(X, \mu, K)$ for $1 \leq p < \infty$ where p and q are related by the equalities

$$p^{-1} + q^{-1} = 1 \quad \Longleftrightarrow \quad pq = p + q \quad \Longleftrightarrow \quad (p-1)(q-1) = 1.$$

More precisely, any functional $F \in L^p(X, \mu, K)^*$ has the form

$$F_g(f) = \int_X f(x)g(x)d\mu(x) \qquad \text{for some } g \in L^q(X, \mu, K)$$

and $\|F_g\|_{L_p(X,\mu,K)^*} = \|g\|_{L_q(X,\mu,K)}$. $\diamondsuit$

The class of all Banach spaces forms a category $\mathcal{B}$ (morphisms are bounded linear operators). The passage from V to V^* can be extended to a contravariant functor from $\mathcal{B}$ to itself. In particular, to any $A \in \mathrm{Hom}(V_1, V_2)$ there corresponds the **dual** or **adjoint** operator $A^* \in \mathrm{Hom}(V_2^*, V_1^*)$ defined

by

$$\langle A^* f,\, x\rangle = \langle f,\, Ax\rangle.$$

One can prove that $\|A^*\| = \|A\|$.

In the finite-dimensional situation the Banach spaces V and V^* play dual roles: the second dual $V^{**} := (V^*)^*$ is canonically isomorphic to V.

This is no longer true in the infinite-dimensional case. In general we have only the map: $V \hookrightarrow V^{**}$. Indeed, to any $v \in V$ we can associate the linear functional F_v on $V^* :\ \langle F_v,\, f\rangle := \langle f,\, v\rangle$. It is known that this map is actually an isometric embedding of V into V^{**}.

Those Banach spaces for which the embedding $V \hookrightarrow V^{**}$ is an isomorphism are called **reflexive**; e.g., the spaces $L_p(X,\, \mu,\, K)$ are reflexive for $1 < p < \infty$.

1.3. Vector integrals.

In Appendix II.2.4 we defined the integral of a density over a smooth manifold.

In representation theory we need the notion of an integral not only for real or complex-valued densities but also for densities ω with values in a given real or complex vector space V. When V is finite-dimensional, we can introduce a basis $\{v_1,\, \dots,\, v_n\}$ and define the integral $\int_M \omega(m)$ as a vector in V whose k-th coordinate is the integral $\int_M \omega^k(m)$ where $\omega^k(m)$ is the k-th coordinate of $\omega(m)$. Of course, we have to check that this definition does not depend on the choice of a basis. We leave this to the reader.

For an infinite-dimensional space V the definition of an integral is more delicate. Let V be a Banach space and ω a V-valued density on a manifold M. Suppose that for any $f \in V^*$ the scalar density $\langle f,\, \omega(m)\rangle$ is integrable over M (e.g., it is so when ω is weakly continuous and has a compact support). Then we can define the **weak integral** w-$\int_M \omega(m)$ as a vector in V^{**} such that

$$\left\langle \text{w-}\int_M \omega(m),\ f \right\rangle = \int_M \langle f,\, \omega(m)\rangle. \tag{4}$$

If we assume that ω is strongly continuous (i.e. locally $\omega = f(x)d^n x$ where f is a strongly continuous vector function) and has a compact support, then the **strong integral** s-$\int_M \omega(m)$ is defined. It is a vector in V which is the limit of Riemannian integral sums constructed exactly as in the scalar case (see Appendix II.2.4):

$$\text{s-}\int_M \omega(m) = \lim S(f; \{M_i\}, \{x_i\}) \ = \lim \sum_i f(x_i) \cdot |vol|(M_i).$$

Since the integrand is strongly continuous and has a compact support, the integral sums form a Cauchy sequence, hence have a limit when the diameter of the partition $M = \bigsqcup_i M_i$ tends to zero.

Exercise 1. Prove that when the strong integral s-$\int_M \omega(m)$ exists, then the weak integral w-$\int_M \omega(m)$ also exists, takes its value in V, and coincides with the strong integral. ♣

1.4. Hilbert spaces.

The infinite-dimensional real and complex Hilbert spaces are the most direct analogues of the ordinary Euclidean space $\mathbb{R}^N$ and its complex version $\mathbb{C}^N$.

They are the very particular cases of Banach spaces for which the non-linear object, the norm, can be expressed in terms of a linear object: the scalar (or inner) product.

By definition, a **scalar product** in a complex vector space V is defined as a sesquilinear[1] map: $V \times V \to \mathbb{C}$, denoted by (x, y), with the following properties:

(i) $(x, y) = \overline{(y, x)}$ (Hermitian symmetry)

(ii) $(x, x) \geq 0$ and $(x, x) = 0 \iff x = 0$ (Positivity)

The **scalar product** in a real vector space is defined in the same way, but its properties are slightly simpler. Namely, the scalar product (x, y) is real and bilinear in x and y. The symmetry condition (i) is just $(x, y) = (y, x)$.

A Banach space V is called a **Hilbert space** if V admits a scalar product such that the norm is related to the scalar product by the equation

$$\|x\| = \sqrt{(x, x)} \quad \text{for all } x \in V. \tag{5}$$

It is known that this is the case if and only if the norm satisfies the so-called **parallelogram identity:**

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Geometrically it means that for any parallelogram the sum of squares of the two diagonals equals the sum of squares of the four sides.

Remark 1. This definition also includes finite-dimensional Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$. Initially, the term Hilbert space was reserved for the space introduced by Hilbert. In modern terms it can be defined as "complex infinite-dimensional separable Hilbert space" or "complex Hilbert space

[1] That is, complex-linear, in the first argument and antilinear, or conjugate-linear, in the second one. In Russian mathematical literature a more expressive term "$\frac{3}{2}$-linear" is used.

of countable Hilbert dimension" (see below). We prefer the more general definition above. ♡

For a vector v in a Hilbert space V the norm is usually denoted by $|v|$ and called the **length** of v. In a real vector space one can also define the **angle** ϕ between two vectors v and w by the equality

$$\cos\phi = \frac{(v,\, w)}{|v|\cdot|w|}, \quad 0 \le \phi \le \pi. \tag{6}$$

Two vectors v and w are called **orthogonal** if $(v,\, w) = 0$. This definition makes sense in complex Hilbert spaces in contrast with the notion of an angle.

For a finite system $\{v_i\}$ of pairwise orthogonal vectors the following equality holds **(Pythagorean Theorem)**:

$$\left|\sum_i v_i\right|^2 = \sum_i |v_i|^2 \tag{7}$$

The system $\{v_\alpha\}_{\alpha\in A}$ of vectors in a Hilbert space V is called **orthonormal** if

$$(v_\alpha,\, v_\beta) = \delta_{\alpha,\beta}.$$

For any such system and for any vector $v \in V$ the quantities $c_\alpha = (v,\, v_\alpha)$ are called **coefficients** of v_α with respect to the system $\{v_\alpha\}_{\alpha\in A}$.

Proposition 1. *The following* **Bessel inequality** *holds:*

$$\sum_{\alpha\in A} |c_\alpha|^2 \le |v|^2. \tag{8}$$

Proof. By definition, the sum on the left-hand side of inequality (8) is $\sup_{A_0\subset A}\sum_{\alpha\in A_0}|c_\alpha|^2$ where A_0 runs over all finite subsets of A. Therefore, it is enough to prove the inequality for a finite set A_0.

In this case we write $v = v' + v''$ where $v' = \sum_{\alpha\in A_0} c_\alpha v_\alpha$, and observe that v and v' have the same coefficients with respect to the system $\{v_\alpha\}_{\alpha\in A_0}$. It follows that v'' is orthogonal to v' and from the Pythagorean Theorem we have

$$|v|^2 = |v'|^2 + |v''|^2 = \sum_{\alpha\in A_0} |c_\alpha|^2 + |v''|^2 \ge \sum_{\alpha\in A_0} |c_\alpha|^2, \tag{9}$$

which proves the Bessel inequality. □

An orthonormal system $\{v_\alpha\}_{\alpha\in A}$ in V is called **complete** if there is no non-zero vector $v \in V$ that is orthogonal to all v_α, $\alpha \in A$. In this case the system forms a **Hilbert basis** in V in the following sense.

Proposition 2. *For any $v \in V$ the series $\sum_{\alpha \in A} c_\alpha v_\alpha$ converges to v.*

In more detail, for any $\epsilon > 0$ there is a finite subset $A(\epsilon) \subset A$ such that

$$\left| v - \sum_{\alpha \in A_1} c_\alpha v_\alpha \right| < \epsilon \qquad \textit{for any } A_1 \supset A(\epsilon).$$

Proof. The proof follows from the analogue of (9). □

Warning. A Hilbert basis is *not a basis in the algebraic sense*: the finite linear combinations of basic vectors do not exhaust the whole space. ♠

It is known that every Hilbert space V has a Hilbert basis and that all these bases in a given space V have the same cardinality. This cardinality is usually called the **Hilbert dimension** and is denoted by $\operatorname{dimh} V$. The Hilbert dimension $\operatorname{dimh} V$ coincides with the algebraic dimension $\dim V$ when the latter is finite.

All Hilbert spaces of given Hilbert dimension are isomorphic, i.e. there exists a linear isometric operator from one space to another.

We shall deal mostly with **separable** Hilbert spaces that contain a countable dense subset. They have finite or countable Hilbert dimension (the latter sometimes is denoted by $\aleph_0$).

The dual space V^* to a Hilbert space V can be naturally identified with the complex conjugate space[2] $\overline{V}$. Namely, any linear functional F on V has the form $F = F_w$ where

$$F_w(v) = (v, w), \qquad w \in V. \tag{10}$$

It is convenient to denote F_w by $\overline{w}$. The correspondence $w \leftrightarrow F_w = \overline{w}$ is antilinear and isometric.

2. Operators in Hilbert spaces

In this section we describe different types of operators in Hilbert space. The most important fact is that some of these operators share the main property of numbers: they can serve as arguments for functions of one variable (see Remark 2 below). This fact is used in the construction of the mathematical model of quantum mechanics (see Section 3).

[2]By definition, $\overline{V}$ is the same real vector space as V, but the multiplication by a complex number λ in $\overline{V}$ is defined as the multiplication by the complex conjugate number $\overline{\lambda}$ in V.

2.1. Types of bounded operators.

Let, as before, $\mathrm{Hom}(V_1, V_2)$ denote the set of all bounded operators from one Hilbert space V_1 to another Hilbert space V_2. This is a Banach space with respect to the operator norm.

The above identification of V_i^* with $\overline{V_i}$ produces the antilinear map:

$$\mathrm{Hom}(V_1, V_2) \to \mathrm{Hom}(V_2, V_1): \ A \mapsto A^*$$

where the **adjoint** operator A^* is defined by

$$(Av_1, v_2)_{V_2} = (v_1, A^*v_2)_{V_1} \qquad \text{for all } v_i \in V_i. \tag{11}$$

If $V_1 = V_2 = V$, we simply write $\mathrm{End}\, V$ instead of $\mathrm{Hom}(V, V)$. This is an associative algebra with the anti-involution $A \mapsto A^*$. The algebraic properties of this anti-involution are:

(12)

$$(A+B)^* = A^* + B^*, \qquad (\lambda A)^* = \overline{\lambda} A^*, \qquad (AB)^* = B^*A^*, \qquad (A^*)^* = A.$$

It is also known that $\|A^*\| = \|A\|$.

Using the anti-involution we can define several important classes of operators in a Hilbert space V.

Hermitian or **self-adjoint** operators S: $\quad S^* = S$;

anti-Hermitian or **skew-Hermitian** operators A: $\quad A^* = -A$;

normal operators N: $\quad NN^* = N^*N$;

unitary operators U: $\quad UU^* = U^*U = 1$;

orthoprojectors P: $\quad P^* = P = P^2$.

Remark 2. The normal operators can be considered as a far-reaching generalization of complex numbers. More precisely, Hermitian operators are analogues of real numbers, anti-Hermitian operators correspond to pure imaginary numbers, and unitary operators correspond to numbers of absolute value 1.

In this approach the theory of unitary representations appears as the natural generalization of the theory of characters for abelian groups. ♡

2.2. Hilbert-Schmidt and trace class operators.

Let V be a Hilbert space, and let $V^* \simeq \overline{V}$ be its dual. Consider the tensor product $E = V \otimes V^*$. This space can be identified with the subspace $\mathrm{End}_0(V) \subset \mathrm{End}(V)$ consisting of all operators of finite rank. Namely, to a vector $v_1 \otimes \overline{v_2} \in V \otimes V^*$ we associate the operator $A_{v_1 \otimes \overline{v_2}} \in \mathrm{Hom}\,(V, W)$ defined by

$$A_{v_1 \otimes \overline{v_2}}(v_3) = (v_3, v_2) \cdot v_1.$$

There is no distinguished norm in $V \otimes V^*$. But still some norms are more natural than others. We consider only so-called **cross-norms** in E that satisfy the conditions:

1. $\|v_1 \otimes \overline{v}_2\|_E = |v_1| \cdot |v_2|$ for $v_1 \in V, \overline{v}_2 \in \overline{V}$,
2. $\|\overline{v}_1 \otimes v_2\|_{E^*} = |v_1| \cdot |v_2|$ for $\overline{v}_1 \in \overline{V},\ v_2 \in V$.

There are three remarkable cross-norms in E:

a) the operator norm $\|\cdot\|$ inherited from $\operatorname{End} V$;

b) the **Hilbert cross-norm** $\|\cdot\|_2$ given by

$$\|A\|_2^2 = \operatorname{tr}(A^*A) = \sum_{i \in I} |Av_i|^2 \tag{13}$$

where $\{v_i\}_{i \in I}$ is any orthonormal basis in V;

c) the so-called **trace norm** $\|\cdot\|_1$. There are three equivalent definitions of this norm:

(14)

$$\|A\|_1 = \inf_{A = \sum_k A_k} \sum \|A_k\|, \quad \operatorname{rk} A_k = 1,$$

$$\|A\|_1 = \sum_k s_k(A) \quad \text{where } s_k(A) \text{ are the eigenvalues of } |A| := \sqrt{A^*A},$$

$$\|A\|_1 = \sup_{\{v_k\},\{w_k\}} \sum_k |(Av_k, w_k)| \quad \text{where } \{v_k\} \text{ and } \{w_k\} \text{ are two bases in } V$$

and supremum is taken over all pairs of bases.

Proposition 3. a) *For any cross-norm p and any $A \in \operatorname{End}_0 V$ we have:*

$$\|A\| \leq p(A) \leq \|A\|_1. \tag{15}$$

b) *There exists a unique cross-norm on E satisfying the parallelogram identity, namely the Hilbert cross-norm $\|\cdot\|_2$.*

c) *If A and B are operators of finite rank in V, then so is $C = BA$ and*

$$\|C\|_1 \leq \|A\|_2 \|B\|_2. \tag{16}$$

□

The completion of E with respect to the operator norm coincides with the closure of $\operatorname{End}_0 V$ in $\operatorname{End} V$, which consists of all **compact operators** in V.

The completion E_2 of E with respect to the Hilbert cross-norm consists of so-called **Hilbert-Schmidt operators** A in V characterized by the property:

$$\|A\|_2^2 = \operatorname{tr} A^*A < \infty.$$

The scalar product in E_2 is given by

$$(A,\, B) = \operatorname{tr} B^* A = \sum_{i \in I} (Av_i,\, Bv_i). \tag{17}$$

The completion E_1 of E with respect to the trace norm consists of all operators of **trace class**, characterized by the condition

$$\sum_{i \in I} |Av_i| < \infty \quad \text{for any Hilbert basis } \{v_i\}_{i \in I} \text{ in } V. \tag{18}$$

It follows from (18) that for a trace class operator A the series $\sum_{i \in I} (Av_i,\, v_i)$ is absolutely convergent for any Hilbert basis $\{v_i\}_{i \in I}$. It turns out that the sum of this series does not depend on the basis. This sum is called the **trace** of A.

Proposition 4. *Let $V = L^2(X,\, \mu)$, and let A be an integral operator*

$$(Af)(x) = \int_X a(x,\, y) f(y) d\mu(y). \tag{19}$$

Then A is of Hilbert-Schmidt class iff the kernel $a(x,\, y)$ belongs to $L^2(X \times X,\, \mu \times \mu)$. Moreover,

$$\|A\|_2^2 = \|a\|^2_{L^2(X \times X,\, \mu \times \mu)}. \tag{20}$$

Proof. The proof follows immediately from (13). □

Proposition 5. *Let M be a smooth compact manifold, let μ be a measure on M given by a smooth density, and let A be an integral operator given by formula* (19)*. Then A is of trace class and*

$$\operatorname{tr} A = \int_M a(x,\, x) d\mu(x). \tag{21}$$

Proof. Let us choose a finite atlas on the manifold $M \times M$ such that the diagonal $\Delta M \subset M \times M$ is covered by charts of the form $U_i \times U_i$ and the rest of $M \times M$ is covered by charts of the form $U_j \times U_k$ where $U_j \cap U_k = \emptyset$. We can also assume that all U_i are bounded domains in $\mathbb{R}^n$ endowed with a unimodular coordinate system.

Using the partition of unity subordinated to $\{U_i\}$, we reduce the problem either to the case when $\operatorname{supp} a \subset U_i \times U_i$ or to the case $\operatorname{supp} a \subset U_j \times U_k$, $U_j \cap U_k = \emptyset$.

In the second case the right-hand side of (21) is evidently zero. The trace of A is also zero, because for an appropriate basis in $L^2(M, \mu)$ all summands (Av_i, v_i) are zeros.

In the first case we can consider U_i as a bounded domain in $\mathbb{R}^n$, hence as a domain in $\mathbb{T}^n$, the n-dimensional torus. So, we have reduced the problem to the case $M = \mathbb{T}^n$. In this case it can be easily solved using Fourier series. Indeed, if we put $v_k(x) = e^{2\pi i(k,x)}$ and $a(x, y) = \sum_{k,l\in\mathbb{Z}^n} a_{k,l}e^{2\pi i((k,x)+(l,y))}$, then

$$\sum_i (Av_i, v_i) = \sum_{k\in\mathbb{Z}^n} a_{k,-k} = \int_{\mathbb{T}^n} a(x, x)dx.$$

□

Remark 3. There are two important extensions of formula (21).

First, it remains true in the case of non-compact manifolds if the kernel function $a(x, y)$ has compact support or rapidly decays at infinity. In particular, it is true for $M = \mathbb{R}^n$ and $a \in \mathcal{S}(\mathbb{R}^{2n})$, the Schwartz space.

Second, this formula is also true for a positive integral operator A with a continuous kernel $a(x, y)$ on an arbitrary manifold with a continuous volume form $d\mu$. Moreover, in this case we have $\|A\|_1 = \operatorname{tr} A$.

Note also that formula (21) for the trace of an operator is the direct analog of the formula for the trace of a matrix: $\operatorname{tr} A = \sum_i A_{ii} = (Av_i, v_i)$. We replace the sum of diagonal elements of A by the integral of the kernel function a over the diagonal. ♡

2.3. Unbounded operators.

The theory of bounded operators in a Hilbert space is rather deep and beautiful, but it does not enclose many natural operators (for instance, differential operators). There is the remarkable theory of unbounded operators in a Hilbert space and we recall here some results from this theory.

By an **unbounded operator** in a Hilbert space V we understand a pair (A, D_A) where D_A is a linear subspace (not necessarily closed) in V and $A : D_A \to V$ is a linear operator. The space D_A is called the **domain of definition** for A.

Often people simply write A instead of the full notation (A, D_A). We also follow this tradition, although sometimes it leads to misunderstandings or even mistakes.

When the Hilbert space in question is $L^2(M, \mu)$ where M is a smooth manifold, μ is a measure defined by a smooth density, and the operator A is a differential operator with smooth coefficients, then there is a so-called

natural domain of definition for A. It consists of all regular distributions $f \in L^2(M, \mu)$ for which Af also is regular and belongs to $L^2(M, \mu)$.

It is convenient to study an unbounded operator A via its **graph** Γ_A:

$$\Gamma_A = \{(v, Av) \in V \oplus V \mid v \in D_A\}. \tag{22}$$

An operator A is called **closed** if its graph Γ_A is a closed subspace in $V \oplus V$.

We say that $\widetilde{A}$ is an **extension** of A if $\Gamma_A \subset \Gamma_{\widetilde{A}}$. In this case $D_A \subset D_{\widetilde{A}}$ and $\widetilde{A}|_{D_A} = A$.

The **adjoint** operator (A^*, D_{A^*}) is defined by

$$\Gamma_{A^*} = I \cdot \Gamma_A^{\perp} \tag{23}$$

where $^{\perp}$ denotes the orthogonal complement:

$$\Gamma_A^{\perp} = \{(x, y) \in V \oplus V \mid (x, v) + (y, w) = 0 \text{ for all } (v, w) \in \Gamma_A\}$$

and I denotes "the rotation on a right angle" in $V \oplus V$:

$$I(x, y) = (-y, x).$$

Warning. This definition makes sense only for operators A with a dense domain of definition. Otherwise, if $0 \neq u \perp D_A$, the vector $(u, 0) \in V \oplus V$ is orthogonal to Γ_A. Hence, $(0, u) \in D_{A^*}$, which is impossible: a linear operator cannot send 0 to a non-zero vector u.

Note, however, that there is a theory of **linear relations** generalizing the notion of linear operators. Any vector subspace in $V_1 \oplus V_2$ is by definition a graph of a linear relation between V_1 and V_2. One can define the inverse relation between V_2 and V_1 and also the composition of relations $A \in \mathrm{Rel}(V_1, V_2)$ and $B \in \mathrm{Rel}(V_2, V_3)$.

In the case of Hilbert space V the adjoint relation A^* is defined for any relation $A \in \mathrm{Rel}(V, V)$. ♠

We leave to the reader the labor and pleasure of convincing yourself that for a bounded operator A the definitions (11) and (23) coincide.

Note also that any adjoint operator is closed since the orthogonal complement is always a closed space.

An unbounded operator A is called **self-adjoint** if it coincides with its adjoint operator A^*. Here the equality $D_A = D_{A^*}$ is also understood.

An operator A is called **essentially self-adjoint** if its closure $\overline{A}$ is a self-adjoint operator. (In this case $\overline{A}$ also coincides with A^*.)

An operator A is called **symmetric** if $A \subset A^*$. This means that $(Av, w) = (v, Aw)$ for all $v, w \in D_A$.

Example 5. Let $H = L^2(\mathbb{R}, |dx|)$, and let A be the operator $i \cdot \frac{d}{dx}$ with $D_A = \mathcal{A}(\mathbb{R})$, the space of smooth complex-valued functions on $\mathbb{R}$ with compact support. This operator is symmetric as one can easily see via integration by parts.

One can check that the adjoint operator A^* is also $i \cdot \frac{d}{dx}$ but with bigger domain of definition. Namely, D_{A^*} is exactly the natural domain of definition. It consists of all functions f with generalized derivative $f' \in L^2(\mathbb{R}, dx)$. In this case we also have $A^{**} = A^*$. So, A^* is self-adjoint and A itself is essentially self-adjoint.

The situation changes if we consider the open interval $(0, \infty)$ or $(0, 1)$ instead of the whole line. The operator $A = i \cdot \frac{d}{dx}$ with the domain $D_A = \mathcal{A}(0, \infty)$ or $D_A = \mathcal{A}(0, 1)$ is still symmetric but not essentially self-adjoint.

In the case of a finite interval it has infinitely many self-adjoint extensions, while for the infinite interval there is no self-adjoint extension at all. $\diamondsuit$

Exercise 2. Show that any symmetric operator A with a dense domain of definition admits a closure.

Hint. Prove that A^* is the closure of A using the following fact.

Lemma 1. *Let S be a not necessarily closed subspace in V. Then $\left(S^\perp\right)^\perp$ coincides with the closure $\overline{S}$.* □

For future use we give here the simple

Criterion. *A symmetric operator (A, D_A) in V is essentially self-adjoint iff*

$$\ker(A^* + i \cdot 1) = \ker(A^* - i \cdot 1) = 0.$$

Proof of the non-trivial part. Assume that $\ker(A^* + i \cdot 1) = \ker(A^* - i \cdot 1) = 0$. For any operator B the following equality holds:

$$(\operatorname{im} B)^\perp = \ker B^*.$$

Therefore, both $V_- := \operatorname{im}(A - i \cdot 1)$ and $V_+ := \operatorname{im}(A + i \cdot 1)$ are dense subspaces in V. Define the operator $U_0 : V_- \to V_+$ by $(Av - iv) \mapsto (Av + iv)$. Since A is symmetric, we have the equality $|Av \pm iv|^2 = |Av|^2 + |v|^2$. It follows that U_0 preserves the length, consequently can be extended to a unitary operator U in V. Hence, $A^* = i(U+1)(U-1)^{-1}$ is self-adjoint. □

2.4. Spectral theory of self-adjoint operators.

It is well known that any Hermitian matrix A can be reduced to a diagonal form by a unitary conjugation: $UAU^{-1} = D$. Moreover, the diagonal

entries $\delta_i = D_{ii}$ of D (the eigenvalues of A) are defined uniquely up to permutation.

So, the full set of invariants for a Hermitian operator in an n-dimensional Hilbert space is the collection of real numbers $\delta_1 < \delta_2 < \cdots < \delta_k$ with multiplicities $n_1, n_2, \ldots, n_k$ subjected to the condition $\sum_{i=1}^{k} n_i = n$.

Another way to encode this information is to associate to a Hermitian operator A a collection of disjoint finite subsets $X_1, X_2, \ldots, X_k$ of $\mathbb{R}$ where X_j is the set of eigenvalues of A with multiplicity j. Clearly, the collection $\{X_j\}$ is subjected to the condition $\sum_{j=1}^{k} j \cdot \# X_j = n$.

Let us consider in more detail the case when $n_i = 1$ for all i. In this case we say that the operator A has a **simple spectrum**.

Recall that a vector $v \in V$ is called a **cyclic vector** for an operator A if the vectors $v, Av, \ldots, A^{n-1}v$ are linearly independent, hence span the whole space $V = \mathbb{C}^n$.

Proposition 6. *The Hermitian operator A in an n-dimensional space has a simple spectrum if and only if it has a cyclic vector.* □

More generally, we say that a Hermitian operator A in an n-dimensional space has a **homogeneous spectrum** of **multiplicity** m if all its eigenvalues have the same multiplicity m.

It is clear that such an operator is just a direct sum of m copies of an operator with a simple spectrum. Of course, this can happen only if m is a divisor of n.

It is also clear that any operator can be uniquely written as a direct sum of operators with disjoint homogeneous spectra.

The situation seems to be quite different in the infinite-dimensional case: a Hermitian operator can have no eigenvectors at all (see examples below). The main achievement of the theory of unbounded operators is the spectral theory of self-adjoint operators. It claims that any self-adjoint (not necessarily bounded) operator in a Hilbert space can be reduced by a unitary conjugation to some canonical form.

Below we give the rigorous formulation, which is rather involved.

We start with a reformulation of the above properties of Hermitian operators in the finite-dimensional case. Let us consider a finite-dimensional complex vector space V as the space $L(X)$ of all complex-valued functions on some finite set X with $\# X = \dim V$. We define the inner product in $L(X)$ as

$$(f_1, f_2) = \sum_{x \in X} f_1(x)\overline{f_2(x)}.$$

The natural orthonormal basis in $L(X)$ consists of functions δ_x, $x \in X$:

$$\delta_x(y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Assume that an operator A has a real diagonal matrix in this basis. Then it is just the multiplication operator $f \mapsto a \cdot f$ where a is a real-valued function on X. We see that any Hermitian operator A in the appropriate "functional" realization of the initial Hilbert space is an operator of multiplication by a real-valued function.

It turns out that this statement remains true in the infinite-dimensional case. More precisely, consider any Hilbert space of the form $V = L^2(X, \mu, \mathbb{C})$ (i.e. the space of square-integrable complex-valued functions f on a set X endowed with a measure μ). Let A be the operator of multiplication by a real-valued measurable function $a \in L^\infty(X, \mu, \mathbb{C})$. Then A is a Hermitian operator in V with the norm

$$\|A\|_V = \|a\|_{L^\infty(X, \mu, \mathbb{C})}.$$

Moreover, if a is any real measurable (not necessarily bounded) function on X, we can consider the unbounded operator A acting as multiplication by a with the domain

$$D_A = \{f \in V \mid af \in V\}.$$

It is not difficult to check that A is a self-adjoint operator in V. The spectral theorem below claims that it is the most general form of a self-adjoint operator in a Hilbert space.

Weak Form of the Spectral Theorem. *Let A be a self-adjoint operator in a complex Hilbert space V. Then there exist an isomorphism $\phi\colon V \to L^2(X, \mu, \mathbb{C})$ and a real measurable function a on X such that $\phi \circ A \circ \phi^{-1}$ is the operator of multiplication by a.* □

This variant of the spectral theorem seems to be less informative than the initial finite-dimensional theorem about the reduction of a Hermitian matrix to a diagonal form. But it is quite useful and sufficient for many purposes. For instance, we derive from this theorem the existence of the so-called **operator calculus**, which allows us to treat Hermitian operators as real numbers and use them as arguments of functions.

Theorem 1. *Let A be a self-adjoint operator in a Hilbert space V. There is a unique homomorphism $\phi \mapsto \phi(A)$ from the algebra of complex-valued Borel*

measurable bounded functions on $\mathbb{R}$ *to the algebra* End V *with the properties:*

(i) $\overline{\phi}(A) = \phi(A)^*$;

(ii) $\|\phi(A)\| \leq \operatorname*{ess\,sup}_{x\in\mathbb{R}} |\phi(x)|$;

(iii) *if* $\psi(x) = x\phi(x)$ *is bounded, then* $\psi(A) = A\phi(A)$;

(iv) *if* $|\phi_n(x)| \leq C$ *and* $\phi_n(x) \to \phi(x)$ *for all* $x \in \mathbb{R}$, *then* $\phi_n(A) \to \phi(A)$ *in the strong operator topology.*

Proof. First, prove the existence. We can assume, due to the weak form of the Spectral Theorem, that $V = L^2(X, \mu, \mathbb{C})$ and A is the operator of multiplication by a real-valued measurable function a on X. Then we can define $\phi(A)$ as the multiplication operator by $\phi\big(a(x)\big)$. Properties (i)–(iv) can be easily verified.[3] This proves the existence.

Now prove the uniqueness. Recall that the set of all Borel measurable functions can be obtained from a set of piecewise constant functions by pointwise limits. So, it is enough to check the uniqueness for functions e_c of the form

$$e_c(x) = \begin{cases} 1 & \text{for } x \leq c, \\ 0 & \text{for } x > c. \end{cases}$$

From (i) and the homomorphism property we conclude that $E_c := e_c(A)$ are orthoprojectors in V. Moreover, the ranges V_c of E_c form an increasing family of subspaces $V_c \subset V$ on which A is bounded from above:

$$(Av, v) \leq c(v, v) \qquad \text{for all } v \in V_c.$$

The reader acquainted with abstract spectral theory will recognize in E_c the **spectral function** of A. The characteristic property of this function is the following integral formula for the operator calculus:

$$\varphi(A) = \int_{\mathbb{R}} \varphi(x) dE_x = \text{s-}\lim_{n\to\infty} \sum_{k\in\mathbb{Z}} \varphi\left(\frac{k}{n}\right)\left(E_{\frac{k}{n}} - E_{\frac{k-1}{n}}\right) \tag{24}$$

for any bounded Borel measurable function ϕ.

So, the uniqueness of the operator calculus is equivalent to the uniqueness of the spectral function of A. But the latter follows from (24). Indeed, if E'_c is another spectral function for A, then

$$E_c = e_c(A) = \int_{\mathbb{R}} e_c(x) dE'_x = E'_c.$$

□

[3] Property (iv) follows from the Lebesgue Dominant Convergence Theorem.

The important consequence of the uniqueness of the operator calculus is the following

Theorem 2. *If a unitary operator U commutes with A (i.e. U preserves D_A and $UA = AU$ on D_A), then it commutes with any function of A.*

Proof. Indeed, the correspondence $\phi \mapsto U\phi(A)U^{-1}$ satisfies conditions (i)–(iv), hence coincides with $\phi \mapsto \phi(A)$. □

We say that two unbounded self-adjoint operators A_1 and A_2 commute if their spectral functions commute:

$$E_{c_1}(A_1)E_{c_2}(A_2) = E_{c_2}(A_2)E_{c_1}(A_1). \tag{25}$$

It is useful to know that any family $\{A_i\}_{i\in I}$ of pairwise commuting self-adjoint operators can be simultaneously reduced to the form of multiplication by a function.

In the finite-dimensional case it means that any family $\{A_i\}_{i\in I}$ of pairwise commuting Hermitian matrices can be simultaneously reduced to the diagonal form.

There exist more precise versions of the Spectral Theorem that generalize the notion of multiplicity for eigenvalues.

Inspired by the finite-dimensional experience, we say that a self-adjoint operator A in a Hilbert space V has a **simple spectrum** if there exists a cyclic vector $v \in V$ for A, i.e. such that the linear span of $v,\ Av,\ \dots,\ A^n v, \dots$ is dense in V.

We say that A has a **homogeneous spectrum** of **multiplicity** m if it is similar to a direct sum of m copies of an operator with a simple spectrum.

For operators with a simple spectrum the following stronger form of the Spectral Theorem holds:

Theorem 3. *Let A be a self-adjoint operator with a simple spectrum in a Hilbert space V. Then there exist a Borel measure μ on $\mathbb{R}$ and an isomorphism $\phi\colon V \to L^2(\mathbb{R}, \mu, \mathbb{C})$ such that the operator $\phi \circ A \circ \phi^{-1}$ is the multiplication by x in the space $L^2(\mathbb{R}, \mu, \mathbb{C})$. The measure μ is defined by the operator A up to equivalence.*[4] □

In the finite-dimensional case the measure μ has finite support X, which is exactly the set of eigenvalues of A and has the property $\#X = \dim V$. The space $L^2(\mathbb{R}, \mu, \mathbb{C})$ is isomorphic to $L(X)$ above.

[4]Recall that two measures μ and ν are **equivalent** if they have the same collection of sets of measure zero. In this case there exists a μ-almost everywhere positive function $\rho =: \frac{d\mu}{d\nu}$ such that $\mu = \rho \cdot \nu$.

The general self-adjoint operator is a direct sum of operators with disjoint homogeneous spectra. Note that in a separable Hilbert space the multiplicity of the spectrum can take the values $1, 2, \dots, \infty$.

Now we can formulate the

Strong Form of the Spectral Theorem. *Let A be a self-adjoint operator in a complex Hilbert space V. Then there exists a family $\{\mu_k\}_{1\leq k\leq\infty}$ of pairwise disjoint Borel measures on $\mathbb{R}$ such that A is equivalent to the direct sum $\bigoplus_{1\leq k\leq\infty} A_k$ where A_k is the direct sum of k copies of the multiplication operator by x in $L^2(\mathbb{R}, \mu_k, \mathbb{C})$.*

All measures μ_k, $1 \leq k \leq \infty$, are defined by the operator A uniquely up to equivalence. □

Remark 4. Note that a convenient form of the operator A_k above is the operator of multiplication by x in the space of vector-functions $L^2(\mathbb{R}, \mu_k, W_k)$ where W_k is any complex Hilbert space of Hilbert dimension k.

Recall that the **disjointness** of μ_k means that there exists a partition of the real line $\mathbb{R} = \bigsqcup_{1\leq k\leq\infty} X_k$ into a family of disjoint Borel subsets $\{X_k\}$ such that $\mu_k(\mathbb{R}\backslash X_k) = 0$. Replacing, if necessary, every μ_k by an equivalent measure, we can assume that $\mu = \sum_{1\leq k\leq\infty} \mu_k$ is a finite measure on $\mathbb{R}$.

If we assume now that all W_k are subspaces of a single space W spanned by the first k basic vectors, then the initial Hilbert space V becomes a part of $L^2(\mathbb{R}, \mu, W)$ consisting of functions satisfying

$$f(x) \in W_k \qquad \text{for } x \in X_k.$$

The operator A will simply be multiplication by x as in Theorem 3.

Thus, the collection of disjoint measures $\{\mu_k\}$ can be replaced by a pair (μ, m) where μ is a measure, called the **spectral measure** of A, and m is the **multiplicity function** for A defined by

$$m(x) = k \quad \text{for } x \in X_k.$$

The pair (μ, m) forms the **spectral data** for the operator A. ♡

2.5. Decompositions of Hilbert spaces.

In the category of Hilbert spaces, besides the usual notion of a direct sum of a family of objects, there exists a more general operation: a continuous sum or direct integral of objects.

The idea is as follows. Let X be a set with a measure μ and suppose that to any $x \in X$ a complex Hilbert space V_x is attached. Denote by V the direct sum $\bigoplus_{x\in X} V_x$. Consider the collection of V-valued functions f on

X satisfying the condition $f(x) \in V_x$ for all $x \in X$. It is a complex vector space. We want to make it a Hilbert space by introducing the inner product

$$(f_1, f_2) = \int_X \big(f_1(x), f_2(x)\big)_{V_x} d\mu(x). \tag{26}$$

To make this definition rigorous, we have to formulate conditions on functions f that ensure the existence of the integral (26). The difficulty is that our functions take values in a rather complicated space V (or, if you wish, take values in different Hilbert spaces V_x for different x).

There are several ways to bypass this difficulty. All of them are essentially equivalent and we choose the most direct way. Namely, we fix the family of Hilbert spaces

$$V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots \subset V_\infty = V, \qquad \operatorname{dimh} V_k = k,$$

and the measurable decomposition

$$X = \bigsqcup_{k=1}^{k=\infty} X_k.$$

Now we consider the complex vector space $\mathcal{V}$ of functions $f : X \to V$ satisfying the following conditions:

(i) f is a measurable map from X to V (i.e. has measurable coefficients in any Hilbert basis);

(ii) $f(x) \in V_k$ for $x \in X_k$;

(iii) f has a finite norm:

$$\|f\| = \sqrt{\int_X |f(x)|_V^2 d\mu(x)} < \infty. \tag{27}$$

The space $\mathcal{V}$ is a Hilbert space with respect to the norm (27). It is sometimes denoted as

$$\mathcal{V} = \int_X V_x \, d\mu(x) \qquad \text{where } V_x = V_k \text{ for } x \in X_k,$$

and is called a **continuous sum** or **direct integral** of V_x, $x \in X$.

An operator A in $\mathcal{V}$ is called **decomposable** if there exists a family of operators $A_x \in \operatorname{End} V_x$ such that

$$(Af)(x) = A_x f(x) \quad \text{for almost all } x \in X. \tag{28}$$

In this case we use the notation

$$A = \int_X A_x \, d\mu(x) \tag{29}$$

and say that A is a continuous sum, or direct integral, of A_x, $x \in X$.

For any set $\mathcal{S} \subset \operatorname{End} V$ we define the **commutant** of $\mathcal{S}$ as the set

$$\mathcal{S}^! = \{A \in \operatorname{End} V \mid AS = SA \text{ for all } S \in \mathcal{S}\}. \tag{30}$$

A **symmetric** (i.e. stable under the anti-involution *) operator algebra $\mathcal{A} \subset \operatorname{End} V$ with a unit is called a **von Neumann algebra** if it satisfies the condition:

$$\mathcal{A}^{!!} := \left(\mathcal{A}^!\right)^! = \mathcal{A}. \tag{31}$$

From this definition it follows that von Neumann algebras always appear in pairs $(\mathcal{A}, \mathcal{A}^!)$.

Exercise 3. Show that $\mathcal{A} = \mathcal{A}^!$ iff $\mathcal{A}$ is a maximal symmetric abelian subalgebra in $\operatorname{End} V$. ♣

The remarkable fact is that the pure algebraic condition (30) is equivalent to some topological conditions.

von Neumann Theorem. *For symmetric operator algebras $\mathcal{A}$ with a unit the following conditions are equivalent:*

a) *$\mathcal{A}$ is a von Neumann algebra;*

b) *$\mathcal{A}$ is weakly closed;*

c) *$\mathcal{A}$ is strongly closed.* □

A von Neumann algebra $\mathcal{A}$ of operators in the space $\mathcal{V} = \int_X V_x d\mu(x)$ is called **decomposable** if there exists a family of von Neumann algebras $\mathcal{A}_x \subset \operatorname{End} V_x$ such that $\mathcal{A}$ consists of all decomposable operators (28) satisfying the condition

$$A_x \in \mathcal{A}_x \qquad \text{for almost all } x \in X. \tag{32}$$

In this case we denote $\mathcal{A}$ by $\int_X \mathcal{A}_x d\mu(x)$ and call it a **direct integral** of algebras $\mathcal{A}_x$, $x \in X$.

Among all decomposable algebras in $\mathcal{V}$ there is a minimal and a maximal one.

The minimal decomposable algebra $\mathcal{D}$ consists of all **diagonalizable** operators Λ which have the form

$$(\Lambda f)(x) = \lambda(x) f(x)$$

where λ is a measurable essentially bounded complex-valued function on X. Thus, $\mathcal{D}_x = \mathbb{C} \cdot 1_{V_x}$. The algebra $\mathcal{D}$ is abelian and, conversely, every abelian von Neumann algebra can be realized as an algebra of diagonalizable operators in a direct integral of Hilbert spaces.

The maximal decomposable algebra $\mathcal{R}$ consists of all decomposable operators. Here $\mathcal{R}_x = \operatorname{End} V_x$.

Theorem 4. *Let $\mathcal{A}$ be a decomposable von Neumann algebra in the space $\mathcal{V} = \int_X V_x \, d\mu(x)$. Then $\mathcal{A}^!$ is also decomposable and $(\mathcal{A}^!)_x = (\mathcal{A}_x)^!$ for almost all $x \in X$.*

For the proof, see, e.g., [**Di2**]. □

We draw several useful corollaries from this important result.

Corollary 1. *Since $\left(\mathbb{C} \cdot 1_{V_x}\right)^! = \left(\operatorname{End} V_x\right)$ and $\left(\operatorname{End} V_x\right)^! = \left(\mathbb{C} \cdot 1_{V_x}\right)$, we get*

$$\mathcal{D}^! = \mathcal{R}, \qquad \mathcal{R}^! = \mathcal{D}.$$

Corollary 2. *Let A be a self-adjoint operator in a Hilbert space V with spectral data (μ, m) (see Remark 4). Then the algebra $\{A\}^!$ is isomorphic to a continuous sum $\int_{\mathbb{R}} \operatorname{Mat}_{m(x)}(\mathbb{C}) d\mu(x)$ where $\operatorname{Mat}_\infty(\mathbb{C})$ denotes the algebra of all bounded operators in a Hilbert space of Hilbert dimension $\aleph_0$.*

In particular, the algebra $\{A\}^!$ is commutative iff the operator A has a simple spectrum.

2.6. Application to representation theory.

Let (π, H) be a unitary representation of a group G. We say that π is a **continuous sum** (or **direct integral**) of representations π_x, $x \in X$, and write

$$\pi = \int_X \pi_x \, d\mu(x) \tag{33}$$

if there is an isomorphism $\alpha : H \to \mathcal{V} = \int_X V_x \, d\mu(x)$ such that

(i) all operators $\widetilde{\pi}(g) := \alpha^{-1} \circ \pi(g) \circ \alpha$ are decomposable operators in $\mathcal{V}$:

$$\widetilde{\pi}(g) = \int_X \widetilde{\pi}_x(g) \, d\mu(x)$$

and

(ii) the correspondence $g \mapsto \widetilde{\pi}_x(g)$ for almost all $x \in X$ is a unitary representation of G equivalent to π_x.

Theorem 5 (Gelfand-Raikov, 1942; see [**Ge**]). *Any unitary representation of a locally compact group in a separable Hilbert space can be written as a direct integral of irreducible representations.*

Proof. Let (π, H) be a unitary representation of a locally compact group G. Let $\mathcal{A}$ be the von Neumann algebra generated by all operators $\pi(g)$, $g \in G$. Choose a maximal symmetric abelian subalgebra $\mathcal{D}$ in the dual algebra $\mathcal{A}^!$. There exists a decomposition of H into a continuous sum $\int_X H_x d\mu(x)$ such that $\mathcal{D}$ is the algebra of diagonalizable operators. Then $\mathcal{A} \subset \mathcal{D}^!$ will be decomposable: $\mathcal{A} = \int_X \mathcal{A}_x d\mu(x)$. In particular, all operators $\pi(g)$, $g \in G$, are decomposable: $\pi(g) = \int_X \pi_x(g) d\mu(x)$.

For locally compact groups one can derive from this that almost all π_x are unitary representations of G in H_x.

Now, since $\mathcal{A}^! \cap \mathcal{D}^! = \mathcal{D}$, the only decomposable operators in $\mathcal{A}^!$ are those that are in $\mathcal{D}$. Hence, $(\mathcal{A}_x)^! = \mathcal{D}_x = \mathbb{C} \cdot 1_{H_x}$ for almost all $x \in X$. This means that almost all representations π_x are irreducible. □

This result, as in the finite-dimensional situation, justifies the intensive study of irreducible representations. But for infinite-dimensional representations of general groups the role of this result is diminished by the lack of the uniqueness of the decomposition in question. We discuss this problem below.

The great achievement in representation theory made in the 1960's is the dichotomy of all topological groups onto so-called **type I** and non-type I groups.

In [**Ki2**] I suggested the term **wild** for the non-type I groups. Accordingly, groups of type I will be called **tame**.

Roughly speaking, the tame groups have a nice representation theory while the wild groups have all possible unpleasant features (see examples in Chapter 4):

1. The topological space $\widehat{G}$ violates even the weakest separation axiom T_0. This means that there exist two distinct points in $\widehat{G}$ such that any neighborhood of one point contains the second.

2. The decomposition of a unitary representation into irreducible components can be non-unique. It can even happen that two different decompositions have no unirreps in common.

3. There exist factor representations of types II and III in the sense of von Neumann (see [**Di2**] or [**Ki2**]).

4. There exist unirreps that possess no generalized characters. Namely, operators $\pi(\phi)$, $\phi \in \mathcal{A}(G)$, are never of trace class unless they vanish.

For tame topological groups the decomposition into irreducible components is unique in the following sense. Let $\widehat{G}$ denote the set of all equivalence classes of unirreps of G. It is a topological space where the neighborhood of the class λ is defined in Chapter 3, Section 4.5.

Proposition 7 (see [**Di2**]). *Any unitary representation π of a tame group G can be written in the form*

$$\pi = \int_{\widehat{G}} m(\lambda) \cdot \pi_\lambda d\mu(\lambda)$$

where μ is a Borel measure on $\widehat{G}$, m is a Borel measurable function that takes values from $\{1, 2, \ldots, \infty\}$, and π_λ is a unirrep of class λ. The measure μ is defined up to equivalence and the function m is defined uniquely μ-almost everywhere. □

3. Mathematical model of quantum mechanics

In this section we give a brief dictionary between quantum mechanics and mathematics. Note, however, that a completely adequate translation of all notions is impossible; e.g. the physical term "quantization" has several (non-equivalent) mathematical translations.

We recommend to the reader to compare this dictionary with Appendix II.3.3 where we describe the mathematical model of classical mechanics.

Physical notions	Mathematical interpretations
phase space	projective space $P(H)$ associated with a complex Hilbert space H
state of the system	an element of $P(H)$, usually represented by a unit vector $\psi \in H$
physical observable	a self-adjoint operator A on H
the value of an observable A in a given state ψ	a random variable with the mean value $(A\psi, \psi)$ and distribution $(E_c(A)\psi, \psi)$
the energy of a system	a non-negative operator E on H
equation of motion (for states)	$\frac{h}{2\pi i}\dot{\psi} = E\psi$ (Schrödinger equation)

equation of motion (for observables)	$\dot{A} = \frac{h}{2\pi i}[E, A]$ (Heisenberg equation)
state where the given observable A has a determined value a	an eigenvector of A with the eigenvalue a
observables that can be simultaneously measured	commuting operators

These rules are more visual when the operator A has the pure point spectrum, i.e. there is an orthonormal basis $\{\psi_k\}_{0\leq k<\infty}$ consisting of eigenvectors, $A\psi_k = a_k\psi_k$. In this case the random variable A in the state ψ takes the values a_k, $0 \leq k < \infty$, with probabilities $p_k = |(\psi, \psi_k)|^2$. Since $|\psi|^2 = 1$, we have $\sum_k p_k = 1$.

Appendix V

Representation Theory

1. Infinite-dimensional representations of Lie groups

Formally, we do not assume any preliminary knowledge about representation theory. But certainly, some acquaintance with finite-dimensional representations of finite groups will be useful.

In this appendix we collect some basic facts and notions from the theory of infinite-dimensional representations of Lie groups that are used in the main part of the book.

1.1. Generalities on unitary representations.

A **unitary representation** of a group G is a pair (π, H) where H is a Hilbert space and $\pi : \mathfrak{g}^* \to \mathcal{U}(H)$ is a homomorphism of G to the group $\mathcal{U}(H)$ of all unitary operators in H.

In other words, $\pi : G \to \mathcal{U}(H)$ is an operator-valued function on G satisfying the multiplicativity property:

$$\pi(g_1g_2) = \pi(g_1)\pi(g_2) \qquad \text{for all } g_1, g_2 \in G.$$

The scalar product in H is denoted by (x, y), or by $(x, y)_H$ if there are other Hilbert spaces under consideration.

If we choose an orthonormal basis $\{x_\alpha\}_{\alpha\in A}$ in H, then $\pi(g)$ is given by a unitary matrix (possibly of infinite size) with entries

$$\pi_{\alpha,\beta}(g) = (\pi(g)x_\alpha,\, x_\beta)\,.$$

More generally, for any two vectors $x,\, y \in H$ we call a **matrix element** of π the complex-valued function on G:

$$\pi_{x,y}(g) = (\pi(g)x,\, y). \tag{1}$$

When G is a topological group, in particular, a Lie group, it is natural to impose some continuity conditions. For finite-dimensional representations there is only one way to do this, since the unitary group $U(N)$ has only one natural topology.

So, we simply require the continuity of all entries $\pi_{\alpha,\beta}(g)$ in any given basis. It implies the continuity of all matrix elements (1) and also the continuity of the matrix-valued function $\pi(g)$.

For infinite-dimensional representations the situation is more delicate. There are several different ways to introduce a topology in the group $\mathcal{U}(H)$. The most common are **weak, strong**, and **uniform** topologies.

Recall that

• weak continuity of π means that for any two vectors $x,\, y \in H$ the matrix element $\pi_{x,y}(g)$ is continuous;

• strong continuity of π means that for any vector $x \in H$ the vector-valued function $g \to \pi(g)x$ is continuous;

• uniform continuity of π means that the operator-valued function $g \to \pi(g)$ is norm-continuous.

In fact, all interesting infinite-dimensional representations of Lie groups are strongly continuous. But very few of them are continuous in the norm topology (see Example 1 below).

On the other hand, for unitary representations the conditions of weak and strong continuity are equivalent as the following theorem shows.

Theorem 1. *If a unitary representation $(\pi,\, H)$ of a topological group G is weakly continuous at the unit point $e \in G$, then it is strongly continuous everywhere.*

Proof. We have to show that for any $g \in G$, any $x \in H$, and any $\epsilon > 0$ there is a neighborhood V of g such that

$$|\pi(g')x - \pi(g)x|^2 < \epsilon \qquad \text{for any } g' \in V.$$

Since π is unitary, the left-hand side can be written as

$$|\pi(g)x|^2 - 2\,\mathrm{Re}\,\big(\pi(g')x,\ \pi(g)x\big) + |\pi(g')x|^2 = 2\,\mathrm{Re}\,\big((1-\pi(g^{-1}g'))x,\, x\big).$$

But we know that (π, H) is weakly continuous at e. Therefore, there is a neighborhood W of e, such that

$$\left| \, ((\pi(h) - 1)x, \, x) \, \right| < \frac{\epsilon}{2} \qquad \text{for } h \in W.$$

So, it is enough to put $V = gW$. □

Example 1. Let π be the natural representation of $G = \mathbb{R}$ in $L^2(\mathbb{R}, \, dt)$:

$$(\pi(a)f)(t) = f(t + a).$$

We claim that π is strongly continuous. Indeed, every function $f \in L^2(\mathbb{R}, \, dt)$ can be approximated in the L^2-norm by a continuous function f_0 with a compact support:

$$\|f - f_0\|_{L^2(\mathbb{R}, \, dt)} < \epsilon, \quad \operatorname{supp} f_0 \subset [-R, \, R].$$

Further, since f_0 is continuous on $[-R, \, R]$, it is uniformly continuous and for some $\delta > 0$ we have $|f_0(t) - f_0(s)|^2 < \frac{\epsilon^2}{2R}$ as soon as $|t - s| < \delta$. Then, for $|a - b| < \delta$ we have

$$\begin{aligned}\|\pi(a)f - \pi(b)f\| \leq \|\pi(a)f - \pi(a)f_0\| + \|\pi(a)f_0 - \pi(b)f_0\| \\ + \|\pi(b)f_0 - \pi(b)f\| < 3\epsilon.\end{aligned}$$

Thus, π is strongly continuous.

On the other hand, we claim that $\|\pi(a) - \pi(b)\| = 2$ for $a \neq b$.

The inequality $\|\pi(a) - \pi(b)\| \leq 2$ is clear. To prove the converse, we define a function f_N, $N \in \mathbb{N}$, by the formula

$$f_N(t) = \begin{cases} \sin \frac{\pi t}{b-a} & \text{for } |t| \leq N|b - a|, \\ 0 & \text{for } |t| > N|b - a|. \end{cases}$$

Then $\pi(a)f_N$ is almost equal to $-\pi(b)f_N$. More precisely,

$$\|\pi(a)f_N\|^2 = \|\pi(b)f_N\|^2 = N|b-a|, \quad \|\pi(a)f_N - \pi(b)f_N\|^2 = (4N-1)|b-a|.$$

Therefore, π is discontinuous in the norm topology. ◇

This simple example can be widely generalized. Namely, G can be any Lie group and the operators $\pi(g)$ can act on functions (or, more generally, on sections of vector bundles with connection) on smooth G-manifolds by diffeomorphisms combined with multiplication by a smooth (operator-valued) function. So locally, using a trivialization, we can write this action in the form (22) (see Proposition 5 below).

All such representations are strongly continuous but discontinuous in the norm topology.

From now on by a **unitary representation of a Lie group** G we mean a continuous homomorphism $\pi : G \to \mathcal{U}(H)$ where $\mathcal{U}(H)$ is endowed with the strong operator topology.

These representations form a category $\mathcal{R}ep(G)$. The objects of $\mathcal{R}ep(G)$ are unitary representations of G; the morphisms from (π, H) to (π', H') are the so-called **intertwining operators** (or **intertwiners** for short) $A\colon H \to H'$ such that for any $g \in G$ the following diagram is commutative:

$$\begin{array}{ccc} H & \xrightarrow{\pi(g)} & H \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A} \\ H' & \xrightarrow{\pi'(g)} & H'. \end{array} \tag{2}$$

The set of all intertwiners forms a complex vector space denoted by $I(\pi, \pi')$, or by $\mathrm{Hom}_G(H, H')$. The dimension of this space is called the **intertwining number** and is denoted by $i(\pi, \pi')$.

Remark 1. The intertwining number is a very important and useful notion in representation theory. It plays the role of a peculiar inner product between two representations. The evident relations[1]

$$i(\pi_1 \oplus \pi_2, \pi) = i(\pi_1, \pi) + i(\pi_2, \pi), \qquad i(\pi_1, \pi_2) = i(\pi_2, \pi_1), \qquad i(\pi, \pi) \geq 0$$

are analogues of linearity, symmetry, and positivity of an inner product.

Moreover, the famous Schur Lemma (see below) claims that for a finite group G the non-equivalent irreducible representations $\pi_1, \pi_2, \ldots, \pi_n$ form an analogue of an orthonormal basis in $\mathcal{R}ep\, G$:

$$i(\pi_k, \pi_j) = \delta_{kj}.$$

This analogy can be made precise if we pass from representations to their characters:

$$\chi_\pi(g) := \mathrm{tr}\, \pi(g).$$

For any finite group G we have

$$i(\pi_1, \pi_2) = \frac{1}{|G|} \sum_{g \in G} \chi_{\pi_1}(g) \overline{\chi_{\pi_2}(g)}.$$

[1] Are they really evident for you? If not, try to prove them using the definitions. (Note that the definition of a direct sum is given below.)

The same formula is true for compact groups if we replace the average over a finite group by integration over a compact group with respect to a normalized invariant measure. ♡

Two representations (π, H) and (π', H') are called **unitarily equivalent** if there is a unitary intertwiner $U \in i(\pi, \pi')$. In the appropriate bases, related by U, these representations are given by the same matrix-valued function. Therefore, in many cases there is no reason to distinguish between π and π'. For instance, all classification problems deal with equivalence classes of representations, not with representations themselves.

Proposition 1. *Two unitary representations of a group G are unitarily equivalent iff they are equivalent objects of the category $\mathcal{R}ep(G)$.*

Proof of the non-trivial part. Let (π, H) and (π', H') be equivalent objects of $\mathcal{R}ep(G)$. Then there exists an invertible intertwiner $A : H \to H'$. The adjoint operator $A^* : H' \to H$ is also an invertible intertwiner. It follows that A^*A is a positive self-adjoint invertible element of $i(\pi, \pi)$. This operator has a unique positive self-adjoint square root $R = \sqrt{A^*A}$. But for any $g \in G$ the operator $R' = \pi(g)^{-1}R\pi(g)$ is also a positive self-adjoint square root from A^*A, since

$$(R')^2 = \pi(g)^{-1}R^2\pi(g) = \pi(g)^{-1}A^*A\pi(g) = A^*A.$$

Hence, $R' = R$ and $R \in i(\pi, \pi)$. It follows that $U := AR^{-1} \in i(\pi, \pi')$. But $U^*U = R^{-1}A^*AR^{-1} = 1$. We see that U is a unitary intertwiner and π is unitarily equivalent to π'. □

Let (π, H) be a unitary representation of a group G. If H has a closed subspace H_1 that is stable under all operators $\pi(g)$, $g \in G$, then the orthogonal complement $H_2 = H_1^\perp$ is also stable under all $\pi(g)$.

Let $\pi_k(g)$ denote the restriction of $\pi(g)$ to H_k, $k = 1, 2$. It is clear that (π_k, H_k) is itself a unitary representation of G. It is called a **subrepresentation** of (π, H).

In the appropriate basis the matrix of $\pi(g)$ acquires the block-diagonal form:

$$\pi(g) = \begin{pmatrix} \pi_1(g) & 0 \\ 0 & \pi_2(g) \end{pmatrix}.$$

In this case we say that (π, H) is (equivalent to) a **direct sum** of (π_1, H_1) and (π_2, H_2) and write $\pi = \pi_1 \oplus \pi_2$.

Exercise 1. Check that $\pi_1 \oplus \pi_2$ is indeed a direct sum in the category $\mathcal{R}ep(G)$.

(See Appendix II for the definition of a direct sum of objects in a category.) ♣

Exercise 2. Find a non-trivial subrepresentation of the representation from Example 1.

Hint. Consider the Fourier transform of the space $L^2(\mathbb{R}, dt)$. ♣

A representation (π, H) is called **irreducible** if it has no non-trivial subrepresentations. In other words, any closed subspace in H that is stable under all operators $\pi(g)$, $g \in G$, is $\{0\}$ or H itself.

Note that H can have non-closed stable subspaces. They are necessarily dense in H provided they are non-zero.

Theorem 2. *For a unitary representation (π, H) of a group G the following are equivalent:*

1. *(π, H) is irreducible.*
2. *$i(\pi, \pi) = \{\lambda \cdot 1, \lambda \in \mathbb{C}\}$.*
3. *Let $k\pi$ be the representation*

$$\left(\overbrace{\pi \oplus \cdots \oplus \pi}^{k \text{ times}}, \overbrace{H \oplus \cdots \oplus H}^{k \text{ times}}\right) \simeq (\pi \otimes 1, H \otimes \mathbb{C}^k).$$

Then any subrepresentation of $k\pi$ has the form $(\pi \otimes 1, H \otimes W)$, $W \subset \mathbb{C}^k$.

Proof. We show that the first two statements follow from the third one for $k = 1, 2$ respectively.

$3_1 \Rightarrow 1$. Any closed stable subspace in $H \simeq H \otimes \mathbb{C}$ is $H \otimes W$ where W can only be $\{0\}$ or $\mathbb{C}$. Hence, π is irreducible.

$3_2 \Rightarrow 2$. We apply statement 3 for $k = 2$ to the graph Γ_A of an intertwiner $A \in i(\pi, \pi)$. This graph is a closed subspace in $H \oplus H$, which is invariant under operators $2\pi(g)$, $g \in G$. Therefore, Γ_A must have the form $H \otimes W$ where W is a subspace in $\mathbb{C}^2$. If W has dimension 0 or 2, $H \otimes W$ cannot be a graph of an operator. If W is 1-dimensional and is spanned by a vector (α, β), then $H \otimes W$ is the graph of the scalar operator $\lambda \cdot 1$, $\lambda = \frac{\beta}{\alpha}$. (If $\alpha = 0$, $H \otimes W$ is not a graph.)

$1 \Rightarrow 2$. Let $A \in i(\pi, \pi)$ be an intertwiner. Then A^* is also an intertwiner. We can write A in the form $A = A_1 + iA_2$ where $A_1 = \frac{A+A^*}{2}$, $A_2 = \frac{A-A^*}{2i}$ are Hermitian intertwiners. So, it is enough to show that every Hermitian intertwiner A is a scalar operator.

Let $E_c(A)$ be the spectral function for A. Since it is unique, all projectors $E_c(A)$ are intertwiners. Therefore the subspaces $H_c = E_c(A)H$ are invariant under the operators $\pi(g)$. But π is irreducible, hence every H_c is either $\{0\}$ or H. Since $E_c(A)$ is increasing and left-continuous, we conclude that there exists $\lambda \in \mathbb{R}$ such that $E_c(A) = \begin{cases} 0 & \text{for } c \leq \lambda, \\ 1 & \text{for } c > \lambda. \end{cases}$ It follows that $A = \lambda \cdot 1$.

$2 \Rightarrow 3$. Let V be a closed subspace in $H \otimes \mathbb{C}^k$. The orthoprojector P to V belongs to $i(k\pi,\, k\pi)$. In the appropriate basis in $H \otimes \mathbb{C}^k$ the operator P is given by a block-matrix of the form

$$P = \begin{pmatrix} p_{11} & \dots & p_{1k} \\ \dots & \dots & \dots \\ p_{k1} & \dots & p_{kk} \end{pmatrix}$$

where $p_{ij} \in i(\pi,\, \pi)$. From statement 2 we get $p_{ij} = a_{ij} \cdot 1$ and $P = 1 \otimes A$ where A is an orthoprojector in $\mathrm{Mat}_k(\mathbb{C})$. Hence, $V = H \otimes W$ with $W = A\mathbb{C}^k$. □

Remark 2. The representations $(\pi,\, H)$, satisfying the first condition of Theorem 2, are called **topologically irreducible**.

If the representation space H has no invariant subspaces (no matter, closed or non), $(\pi,\, H)$ is called **algebraically irreducible**.

The representation $(\pi,\, H)$, satisfying the second condition of Theorem 2, is called **operator irreducible**.

The representations $(\pi,\, H)$, satisfying the third condition of Theorem 2, are sometimes called **k-irreducible**. This notion makes sense also for $k = \infty$.

For unitary representations ∞-irreducibility is equivalent to k-irreducibility for any finite k.

It is known that for non-unitary representations all the conditions of Theorem 2 are non-equivalent. The simplest example is the group $T_n(\mathbb{C})$ of upper-triangular matrices acting in $\mathbb{C}^n$. There are many invariant subspaces, but no intertwiners except scalar operators. ♡

1.2. Unitary representations of Lie groups.

Let $(\pi,\, H)$ be a unitary representation of a Lie group G. The matrix elements of $(\pi,\, H)$ are bounded and continuous, but not necessarily smooth functions on G.

Example 2. Keep the notation of Example 1 and put $x = y = \chi_{[0,1]}(t)$, the characteristic function of the unit interval. Then

$$\pi_{x,y}(a) = \begin{cases} 0 & \text{if } \ |a| \geq 1, \\ 1 - |a| & \text{if } \ |a| \leq 1. \end{cases}$$

♢

A vector $x \in H$ is called **smooth** if the vector-function $g \mapsto \pi(g)x$ from G to H is strongly infinitely differentiable.

The set of all smooth vectors in H is denoted by H^∞.

Theorem 3. *The vector $x \in H$ is smooth iff for any vector $y \in H$ the matrix element $\pi_{x,y}$ is a smooth function on G.*

The claim results from the following more general statement.

Proposition 2. *Let M be a smooth manifold, let H be a Hilbert space, and let $f \colon M \to H$ be a strongly continuous vector-function. Assume that f is weakly smooth on M, i.e. for any $y \in H$ the scalar function $m \to (f(m), y)$ is smooth. Then f is strongly smooth on M.*

Proof. Since the statement is local, it does not depend on the manifold (all n-dimensional manifolds are locally isomorphic). Therefore, we can assume that $M = \mathbb{T}^n$, the n-dimensional torus. We represent it as $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

We define the **Fourier coefficients** $\{c_k\}$, $k \in \mathbb{Z}^n$, for f as vectors in H given by

$$c_k = \int_{\mathbb{T}^n} e^{-2\pi ikt} f(t) d^n t. \tag{3}$$

Here $t = (t_1, \ldots, t_n)$, $k = (k_1, \ldots, k_n)$, $kt = \sum_i k_i t_i$, $d^n t = |dt_1 \wedge \cdots \wedge dt_n|$, and the integral is defined as the strong limit of Riemann integral sums (see Appendix II.2.4).

For any $y \in H$ the quantities (c_k, y) are Fourier coefficients of the scalar function $t \to (f(t), y)$. Since f is weakly smooth, this function is smooth. Hence, for any $N \in \mathbb{N}$ there is a constant $c(N, y)$ such that

$$|(c_k, y)| \cdot (1 + |k|)^N \le c(N, y).$$

But in a Hilbert space any weakly bounded set is strongly bounded, so for any $N \in \mathbb{N}$ there exists a constant $c(N)$ such that

$$|c_k| \cdot (1 + |k|)^N \le c(N).$$

It follows that f can be written as a strongly convergent series with rapidly decaying coefficients:

$$f(t) = \sum_{k \in \mathbb{Z}^n} c_k \, e^{2\pi ikt}.$$

Therefore, f is a strongly smooth function. □

It is not clear *a priori* if a given representation (π, H) of a Lie group G has at least one non-zero smooth vector. The remarkable fact is that the space H^∞ of smooth vectors is always dense in H. Moreover, it is big enough to reconstruct the representation (π, H) of a connected Lie group G from the representation of the Lie algebra $\mathfrak{g}$ in H^∞.

The main result of this section is

Theorem 4. *Let $(\pi,\, H)$ be a unitary representation of a Lie group G. Then*

1. *The subspace H^∞ of smooth vectors is dense in H and stable with respect to all operators $\pi(g)$, $g \in G$.*

2. *For any $X \in \mathfrak{g} = \mathrm{Lie}\,(G)$ the operator*

$$A = -i\pi_*(X) =: -i\frac{d}{dt}\pi(\exp tX)\bigg|_{t=0} \tag{4}$$

with the domain H^∞ is essentially self-adjoint.

3. *If G is connected, the representation $(\pi,\, H)$ is completely determined by the representation π_* of $\mathfrak{g}$ in H^∞ defined by (4). In particular, for any $x \in H^\infty$ we have*

$$\pi(\exp tX)x \;=\; e^{itA}x \tag{5}$$

where the right-hand side is defined as the solution to the ordinary differential equation

$$x'(t) = \; iAx(t)$$

with the initial condition $x(0) = x$.

The proof needs some preparations.

We recall that to any $X \in \mathfrak{g}$ there correspond two vector fields on G: the left-invariant field R_X (infinitesimal right shift) and the right-invariant field L_X (infinitesimal left shift) given by

$$(R_X f)\,(g) := \frac{d}{dt}f(g\exp tX)\bigg|_{t=0}, \qquad (L_X f)\,(g) := \frac{d}{dt}f(\exp -tX)g\bigg|_{t=0}.$$

For any function $\phi \in \mathcal{A}(G)$ we define the operator $\pi(\phi)$ on H by the formula

$$\pi(\phi) = \int_G \phi(g)\pi(g)d_lg \quad \text{where } d_lg \text{ is a left-invariant measure on } G. \tag{6}$$

Lemma 1. *The following relation holds:*

$$\pi_*(X)\pi(\phi) = \pi(L_X\phi). \tag{7}$$

Proof. Using the left-invariance of $d_l g$, we obtain

$$\pi_*(X)\pi(\phi)x = \int_G \frac{d}{dt}\pi(\exp tX)\phi(g)\pi(g)x d_l g\bigg|_{t=0}$$

$(g \mapsto \exp(-tX)g)$

$$= \int_G \frac{d}{dt}\phi(\exp(-tX)g)\pi(g)x d_l g\bigg|_{t=0}$$

$$= \int_G (L_X\phi)(g)\pi(g)x d_l g = \pi(L_X\phi)x.$$

□

Exercise 3. Show that on H^∞ the following relation holds:

$$\pi(\phi)\pi_*(X) = -\pi\big((R_X + \operatorname{tr}\operatorname{ad} X)\phi\big).$$

Hint. Use the relation $d_l(g\exp -tX) = \Delta(\exp -tX)d_l(g)$.

Proof of Theorem 4. 1. From Lemma 1 it follows that for any $\phi \in \mathcal{A}(G)$ and any $x \in H$ the vector $\pi(\phi)x$ is smooth. Therefore, we sometimes call $\pi(\phi)$ a **smoothing operator**. We have to prove that the linear span of the images of all smoothing operators $\pi(\phi)$, $\phi \in \mathcal{A}(G)$, is dense in H.

Let x be any vector from H. Since π is strongly continuous, for any $\epsilon > 0$ there exists a neighborhood U of $e \in G$ such that $|\pi(g)x - x| < \epsilon$ for all $g \in U$. Take a function $\phi \in \mathcal{A}(G)$ with a support in U such that $\int_U \phi(g)d_l g = 1$, and put $y = \pi(\phi)x$. Then $y \in H^\infty$ and

$$|y - x| = \left|\int_U \phi(g)(\pi(g)x - x)d_l g\right| \le \epsilon \cdot |x|.$$

2. For $X \in \mathfrak{g}$ consider the operator $A = -i\pi_*(X) = -i\frac{d}{dt}\pi(\exp tX)\big|_{t=0}$ with H^∞ as the domain of definition. From the unitarity of $\pi(g)$ we derive that A is symmetric: $(Ax, y) = (x, Ay)$ for all $x, y \in H^\infty$. To check that A is essentially self-adjoint, we use the criterion from Appendix IV.2.3.

Suppose that $y \in \ker(A^* \pm i \cdot 1)$ and consider the function

$$f(t) = (\pi(\exp tX)x, y), \qquad x \in D_A.$$

We have

$$f'(t) = (iA\pi(\exp tX)x, y) = (\pi(\exp tX)x, -iA^*y) = \mp f(t).$$

Hence $f(t) = ce^{\mp t}$. But $f(t)$ is bounded, therefore $c = 0$ and y is orthogonal to D_A. Since D_A is dense in H, the vector y must be zero. So, $\ker(A^* \pm i \cdot 1) = 0$ and the operator iA is essentially self-adjoint.

3. The relation (5) follows from the very definition of e^{itA}. Now, we have to reconstruct (π, H) from (π_*, H^∞). Recall that for any self-adjoint operator $\bar{A}$ the operator $e^{it\bar{A}}$ can be defined as follows. In the appropriate realization of H in the form $L^2(X, \mu)$ the operator $\bar{A}$ is just the multiplication by a real-valued function $a(x)$. Then we define $e^{it\bar{A}}$ as the multiplication by $e^{ita(x)}$.

When $\bar{A}$ is the closure of A, this definition coincides with (4) on the subspace $D_A = H^\infty$. Therefore $\pi(\exp tX)$ coincides with e^{itA} on H^∞ and with $e^{it\bar{A}}$ on the whole space H. So, we know the operators $\pi(g)$ in a neighborhood of the unit element covered by the exponential map. But G, being connected, is generated by any neighborhood of the unit (see Corollary to Lemma 3 in Appendix III.1.3). □

Remark 3. The idea to use smoothing operators was first used by I. M. Gelfand for 1-parametric groups. It was soon applied by Gårding for general Lie groups. Therefore, the subspace in H^∞ spanned by the vectors of the form $\pi(\phi)v$, $v \in H$, is called the **Gelfand-Gårding space**. This space often coincides with H^∞ (see examples below) but I do not know if it is always true. ♢

Remark 4. Theorem 4 gives a non-trivial and important result even in the case of a finite-dimensional representation. In the finite-dimensional case the space H^∞ can be dense in H only if $H^\infty = H$.

So, any unitary finite-dimensional representation of a Lie group is automatically smooth. Actually, it is true for all continuous finite-dimensional representations without the unitarity restriction.

You can try to find an independent proof of this statement in the simplest case $G = \mathbb{R}$. It looks as follows:

Proposition 3. *Let f be a complex-valued function of a real variable t satisfying the conditions:*

1) *f is continuous;*

2) *$f(t+s) = f(t)f(s)$ for all $s, t \in \mathbb{R}$.*

Then f is smooth. ♢

One more corollary from Theorem 4 and its proof is

Stone's Theorem. *Let $U(H)$ be the group of all unitary operators in a Hilbert space H. Any strongly continuous 1-parametric subgroup in $U(H)$ has the form $u(t) = e^{itA}$ where A is a self-adjoint (not necessarily bounded) operator in H.* □

1.3. Infinitesimal characters.

Let $U(\mathfrak{g})$ be the universal enveloping algebra for the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, and let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$ (see Appendix III.1.4). If (π, H) is a unitary representation of a Lie group G, we have a representation of $U(\mathfrak{g})$ in H^∞ that extends the representation π_* of $\mathfrak{g}$. We use the same notation π_* for this representation (sometimes we simply write π when it causes no confusion).

Theorem 5. *Assume that (π, H) is an irreducible unitary representation of a Lie group G. Then for any $A \in Z(\mathfrak{g})$ the operator $\pi_*(A)$ in H^∞ is scalar:*

$$\pi_*(A) = I_\pi(A) \cdot 1_{H^\infty}. \tag{8}$$

The map $I_\pi : Z(\mathfrak{g}) \to \mathbb{C}$ is an algebra homomorphism, called the **infinitesimal character** *of (π, H).*

Proof. Consider the anti-involution * of $U(\mathfrak{g})$ that sends an element $X \in \mathfrak{g}$ to $-X$ and the product $X_1 \cdots X_k$ to $(-1)^k X_k \cdots X_1$. The elements $A \in U(\mathfrak{g})$ satisfying $A^* = A$ are called **Hermitian**.

Since the center $Z(\mathfrak{g})$ is stable under this involution, any element $A \in Z(\mathfrak{g})$ can be written as $A = A_1 + iA_2$ where A_1, A_2 are Hermitian elements of $Z(\mathfrak{g})$. Indeed, it is enough to put $A_1 = \frac{A+A^*}{2}$, $A_2 = \frac{A-A^*}{2i}$. Thus, in the proof of the theorem we can assume that A is Hermitian.

In this case the operator $\pi(A)$ with domain H^∞ is symmetric. Hence, it admits a closure $\overline{\pi(A)}$. The graph of $\overline{\pi(A)}$ is a closed subspace in $H \oplus H$ that is invariant under all operators $(2\pi)(g) = \pi(g) \oplus \pi(g)$. We already have seen in the proof of Theorem 2 that such operators must be scalar. □

In practice, the operators $\pi(g)$, $g \in G$, are usually combinations of shifts and multiplications by a function. Therefore, operators $\pi_*(A)$, $A \in U(\mathfrak{g})$, are differential operators. The decomposition of (π, H) into irreducible components implies the decomposition of H into common eigenspaces for all operators $\pi_*(A)$, $A \in Z(\mathfrak{g})$.

Theorem 5 admits an important generalization. Namely, let R be a G-invariant rational function on $\mathfrak{g}^*$. In Chapter 1, Section 3.1, we showed that such a function can be written as a ratio P/Q of two relatively invariant polynomials of the same weight.

The argument that we used in the proof of Theorem 4 shows that the operators $\pi(\mathbf{sym}(P))$ and $\pi(\mathbf{sym}(Q))$ are proportional. Hence, the infinitesimal character I_π is defined for the extension of $Z(\mathfrak{g})$ that consists of all rational functions on $\mathfrak{g}^*$ that are constant along coadjoint orbits.

1.4. Generalized and distributional characters.

In the theory of finite-dimensional representations the central role is played by the notion of a character. Recall that a **character** of a representation (π, V) of a group G is the scalar function on G given by the formula

$$\chi_\pi(g) = \operatorname{tr} \pi(g).$$

For infinite-dimensional unitary representations this notion no longer makes sense: unitary operators are not in the trace class. Nevertheless, using the smoothing technique, we can define for some infinite-dimensional representations (π, H) the character of π as a distribution or as a generalized function on G.

Namely, assume that for a certain class of test densities μ on G the operators

$$\pi(\mu) = \int_G \pi(g) d\mu(g)$$

are of trace class. Then we can define the **generalized character** χ_π as the generalized function on G, i.e. as a linear functional on the space of test densities given by the formula

$$\langle \chi_\pi, \mu \rangle = \operatorname{tr} \pi(\mu). \tag{9}$$

For Lie groups it is natural to take as test densities the expressions $\mu = \phi(g)d_lg$ where $\phi \in \mathcal{A}(G)$ and $d_l(g)$ is a left-invariant measure on G. (It is clear that using the right-invariant measure $d_r(g)$ we get the same class of test densities.)

Then the right-hand side of (9) can be viewed as a linear functional on $\mathcal{A}(G)$, i.e. a distribution on G. We call it the **distributional character** of π and denote it by χ_π, the same as the ordinary and generalized characters.

It turns out that for some types of Lie groups the generalized or distributional characters indeed exist for all irreducible unitary representations.

They also inherit some important properties of ordinary characters.

Theorem 6. *The generalized (distributional) characters have the properties:*

a) *they are invariant under inner automorphisms of G;*[2]

b) *if π_1 is equivalent to π_2, then $\chi_{\pi_1} = \chi_{\pi_2}$;*

c) *if π_1 and π_2 are irreducible and $\chi_{\pi_1} = \chi_{\pi_2} \not\equiv 0$, then π_1 is equivalent to π_2.*

[2] For distributional characters this property holds only for unimodular groups.

Proof. a) The action of an inner automorphism $A(h)$ on a generalized function or a distribution is defined as a dual to the action on test measures or test functions. Therefore, the statement follows from the invariance of the trace under conjugation: $\operatorname{tr}\pi(A(h)\phi) = \operatorname{tr}\left(\pi(h)\pi(\phi)\pi(h)^{-1}\right) = \operatorname{tr}\pi(\phi)$.

b) If U is an intertwiner for π_1, π_2, then $\operatorname{tr}\pi_2(\phi) = \operatorname{tr}\left(U\pi_1(\phi)U^{-1}\right) = \operatorname{tr}\pi_1(\phi)$.

c) For the proof, which is rather involved, we refer to [**Di3**]. Note that even for finite groups the standard proof actually reconstructs from χ_π not π itself but $d \cdot \pi$ where $d = \dim \pi$. So, in our case we need the analog of Theorem 2 for $k = \infty$ (cf. Remark 2). □

1.5. Non-commutative Fourier transform.

The notion of a Fourier transform is very useful for commutative groups (especially for groups $\mathbb{T}$, $\mathbb{Z}$, $\mathbb{R}$). It turns out that it can be extended to non-commutative groups too. This extension is less simple and less useful but still can help in situations involving a non-commutative group of symmetries.

To define this transform for a given topological group G we have to choose a representative π_λ for any equivalence class $\lambda \in \widehat{G}$. Let dg denote a left-invariant measure on G, which is known to exist on any locally compact group.

For a function $f \in L^1(G, dg)$ we define its Fourier transform $\widehat{f}$ as an operator-valued function $\widehat{f}$ on $\widehat{G}$ given by the formula

$$\widehat{f}(\lambda) = \int_G f(g)\pi_\lambda(g)dg. \tag{10}$$

This transform shares with its commutative prototype the following basic property: it sends the convolution to the ordinary product:

$$\widehat{f_1 * f_2} = \widehat{f_1} \cdot \widehat{f_2}. \tag{11}$$

Note that on a non-commutative group G the convolution is non-commutative. Therefore it cannot be realized by the ordinary multiplication of scalar functions. But for the operator-valued functions the ordinary multiplication is also non-commutative!

For tame groups G the function f can be reconstructed from its Fourier transform via the formula

$$f(g) = \int_{\widehat{G}} \operatorname{tr}\left(\widehat{f}(\lambda)\pi_\lambda(g)^*\right) d\mu(\lambda) \tag{12}$$

where μ is a special measure on $\widehat{G}$ called the **Plancherel measure**, dual to the measure dg on G.

It is known that for a compact group G with the normalized measure dg the Plancherel measure is just the counting measure on $\widehat{G}$ (which is a discrete set) multiplied by $\dim \pi_\lambda$. For non-compact Lie groups the computation of the Plancherel measure is one of the deepest problems of harmonic analysis.

We finish this section with the non-commutative generalization of the Pontryagin duality principle. Denote by $L^2(\widehat{G},\, \mu)$ the Hilbert space of operator-valued functions ϕ on $\widehat{G}$ taking at a point λ a value in $\mathrm{End}(V_\lambda)$ with the norm

$$|\phi|^2 = \int_{\widehat{G}} \mathrm{tr}\,\big(\phi(\lambda)\cdot\phi(\lambda)^*\big)d\mu(\lambda).$$

More accurately, this is a continuous sum over $(\widehat{G},\, \mu)$ of Hilbert spaces formed by Hilbert-Schmidt operators in V_λ (see Appendix IV.2.5).

Non-commutative Plancherel formula. *Let G be a tame locally compact group. The non-commutative Fourier transform can be extended to a unitary isomorphism between $L^2(G,\, dg)$ and $L^2(\widehat{G},\, \mu)$, i.e.*

$$(13)\quad \int_G |f(g)|^2 dg = |f|^2_{L^2(G,\,dg)} = |\widehat{f}|^2_{L^2(\widehat{G},\,d\mu)} = \int_{\widehat{G}} \mathrm{tr}\,\big(\widehat{f}(\lambda)\cdot\widehat{f}(\lambda)^*\big)d\mu(\lambda).$$

2. Induced representations

We start with a brief description of the theory of induced representations for finite groups. We use this "toy model" to introduce the basic notions and notation that will be used later in the case of infinite-dimensional representations of Lie groups.

2.1. Induced representations of finite groups.

Let G be a finite group, and let $H \subset G$ be a subgroup. For any finite-dimensional representation $(\rho,\, W)$ of H we construct a representation of the group G, which is called the **induced representation** and is denoted by $\mathrm{Ind}\,_H^G(\rho,\, W)$. The space where the induced representation acts is denoted by $L(G,\, H,\, \rho,\, W)$. It consists of W-valued functions ϕ on G satisfying

$$(14)\qquad \phi(hg) = \rho(h)\phi(g) \qquad \text{for all } h \in H,\ g \in G.$$

The group G acts on this space by right shifts:

$$(15)\qquad \big((\mathrm{Ind}\,_H^G\rho)(g)\phi\big)\,(g_1) = \phi(g_1 g).$$

Note that right and left shifts on G commute; therefore the operators (15) preserve the condition (14).

The geometric meaning of an induced representation is especially clear when (ρ, W) is the trivial representation $(1, \mathbb{C})$. In this case the space $L(G, H, 1, \mathbb{C})$ is just the space $L(X)$ of complex-valued functions on the right coset space $X = H\backslash G$. More precisely, to a function $\phi \in L(G, H, 1, \mathbb{C})$ we associate the function $f \in L(X)$ according to the rule $f(x) = \phi(g)$ where g is any representative of the coset $x \in X$. From (10) we see that the value $\phi(g)$ does not depend on the choice of a representative.

The action of G in $L(X)$ is just the geometric action by right shifts:

$$((\operatorname{Ind}_H^G \rho)(g)f)(x) = f(x \cdot g). \tag{16}$$

In the general case we interpret the elements $\phi \in L(G, H, \rho, W)$ as sections of some G-vector bundle E over X with the fiber W.

Recall that from the algebraic point of view the space of sections $\Gamma(E, X)$ is a projective module over the algebra of functions $L(X)$. In our case the space $L(G, H, \rho, W)$ carries a natural $L(X)$-module structure: $(f \cdot \phi)(g) = f(Hg)\phi(g)$.

Actually, for finite groups the bundle in question is always trivial. In algebraic terms it means that $L(G, H, \rho, W)$ is a free $L(X)$-module: $L(X, W) \simeq L(X) \otimes W$. To establish it, we choose for any $x \in X$ a representative $s(x)$ of the coset x.

The map $s : X \to G$ is called a **section** of the natural projection $p : G \to X : p(g) = Hg$, if $p \circ s = \mathrm{Id}$.

It is convenient to choose e as a representative of the coset H, i.e. put $s(H) = e$. Later we always assume this condition.

Using a section s we can identify the set G with the direct product $H \times X$. Namely, any element $g \in G$ can be uniquely written in the form

$$g = hs(x), \qquad h \in H,\ x \in X. \tag{17}$$

Indeed, from (17) we derive $x = Hg,\ h = gs(Hg)^{-1}$.

We can now rewrite the formula for the induced representation in another form that is often more convenient. To this end we introduce the so-called

Master equation:

$$s(x)g = h(x, g)s(y). \tag{18}$$

Here $x \in X$ and $g \in G$ are given, $y = x \cdot g$, and $h(x, g) \in H$ is determined by the equation (18).

Proposition 4. a) *The map $h(x, g)$ from $X \times G$ to H satisfies the following* **cocycle equation***:*

$$h(x, g_1 g_2) = h(x, g_1)h(x \cdot g_1, g_2). \tag{19}$$

b) *Let us identify* $\phi \in L(G, H, \rho, W)$ *with* $f \in L(X, W)$, *using the section* $s:\ f(x) = \phi\big(s(x)\big)$. *Then the induced representation* (11) *takes the form*

$$\big((\operatorname{Ind}{}_H^G\rho)(g)f\big)\,(x) = A(x,\,g)f(x\cdot g) \tag{20}$$

where the operator-valued function $A(x,\,g)$ *is defined by*

$$A(x,\,g) = \rho(h(x,\,g)). \tag{21}$$

Proof. a) is an immediate consequence of the Master equation applied to $g = g_1g_2$.

To prove b) we use the relation between f and ϕ and the Master equation. □

Note that (20) is a natural generalization of the geometric representation (16). The remarkable fact is that the converse is also true: all representations of the form (20) are in fact induced representations.

Proposition 5. *Let* G *be a finite group, let* $H \subset G$ *be a subgroup, and let* $X = H\backslash G$ *be the corresponding right coset space. Assume that* π *is a representation of* G *in the space of vector-functions* $L(X,\,W)$ *acting by the formula*

$$\big(\pi(g)f\big)(x) = A(x,\,g)f(x\cdot g) \tag{22}$$

where A *is some operator-valued function on* $X\times G$. *Then there exists a representation* (ρ, W) *of* H *such that* $(\pi, L(X, W))$ *is equivalent to* $\operatorname{Ind}{}_H^G(\rho, W)$.

Proof. From the multiplicative property $\pi(g_1g_2) = \pi(g_1)\pi(g_2)$ we deduce that $A(x,\,g)$ satisfies the cocycle equation

$$A(x,\,g_1g_2) = A(x,\,g_1)A(x\cdot g_1,\,g_2). \tag{23}$$

Lemma 2. *All solutions to the cocycle equation* (23) *have the form*

$$A(x,\,g) = C(x)^{-1}\rho\big(h(x,\,g)\big)C(x\cdot g) \tag{24}$$

where C *is an invertible operator-valued function on* X, $h(x,\,g) \in H$ *is defined by the Master equation, and* ρ *is a representation of* H *in* W.

Proof. Put $g_1 = s(x)$, $x = x_0 := (H)$, and $g_2 = g$ in (23). We obtain

$$A\bigl(x_0,\, s(x)g\bigr) = A\bigl(x_0,\, s(x)\bigr)A(x,\, g) \quad \text{or} \quad A(x,\, g) = B\bigl(s(x)\bigr)^{-1}B\bigl(s(x)g\bigr)$$

where $B(g) := A(x_0,\, g)$.

If we now put $x = x_0$, $g_1 = h$, and $g_2 = g$ in (23), we get $B(hg) = B(h)B(g)$. Finally, putting $C(x) = B\bigl(s(x)\bigr)$ and $\rho(h) = B(h)$ we obtain the desired result. □

To complete the proof of Proposition 5 it is now enough to make the transformation $f(x) \mapsto C(x)f(x)$ that sends $A(x,\, g)$ to the desired form (21) (and the representation (22) to an equivalent representation). □

Now we prove a very useful criterion: when a representation $(\pi,\, V)$ of a finite group G can be written in the form (22). In other words, we want to know when $(\pi,\, V)$ is equivalent to $\mathrm{Ind}\,_H^G(\rho,\, W)$ for some representation $(\rho,\, W)$ of the subgroup $H \subset G$. To formulate this criterion we need some new notions and notation.

Let $X = H\backslash G$ be the set of right H-cosets in G. Denote by $\mathcal{A}(X)$ the collection of all complex-valued functions on X. From the algebraic point of view $\mathcal{A}(X)$ is a commutative associative algebra (with respect to the ordinary multiplication) with an involution $f \mapsto \bar{f}$ (complex conjugation).

Let us define a **$*$-representation** $(\Pi,\, V)$ of $\mathcal{A}(X)$ as a map $\Pi : \mathcal{A}(X) \to \operatorname{End} W$, which is an algebra homomorphism with an additional property $\Pi(\bar{f}) = \Pi(f)^*$ where $*$ means Hermitian adjoint operator.

We call a $*$-representation $(\Pi,\, V)$ **non-degenerate** if $\Pi(1) = 1$. Here 1 in the left-hand side is the constant function on X with value 1, and 1 in the right-hand side denotes the unit operator in V.

Let us say that a $*$-representation $(\Pi,\, V)$ of $\mathcal{A}(X)$ and a unitary representation $(\pi,\, V)$ of G in the same space are **compatible** if

$$\pi(g)\Pi(\phi)\pi(g)^{-1} = \Pi\bigl(R(g)\phi\bigr) \quad \text{where } \bigl(R(g)\phi\bigr)(x) = \phi(x \cdot g). \tag{25}$$

To any representation of the form (18) we can easily associate a $*$-representation $(\Pi,\, V)$ of $\mathcal{A}(X)$ compatible with $(\pi,\, V) = (\mathrm{Ind}_H^G(\rho, W),\, L(X, W))$. Namely, put

$$\bigl(\Pi(\phi)f\bigr)(x) = \phi(x)f(x). \tag{26}$$

Then

$$\begin{aligned}\bigl(\pi(g)\Pi(\phi)f\bigr)(x) &= A(x,\, g)\phi(x \cdot g)f(x \cdot g)\\ &= \phi(x \cdot g)A(x,\, g)f(x \cdot g) = \bigl(\Pi\bigl(R(g)\phi\bigr)\pi(g)f\bigr)(x)\end{aligned}$$

and (25) is satisfied. Note that this representation Π is non-degenerate.

Inducibility Criterion. *A unitary representation* (π, V) *of a finite group* G *is induced from some representation* (ρ, W) *of a subgroup* H *iff there exists a non-degenerate* $*$*-representation* (Π, V) *of* $\mathcal{A}(H\backslash G)$ *compatible with* (π, V).

Proof. We have seen above that the condition is necessary. We prove that it is also sufficient. Let $X = H\backslash G$. We introduce in $\mathcal{A}(X)$ the special basis consisting of functions δ_a, $a \in X$, where

$$\delta_a(x) = \begin{cases} 1 & \text{for } x = a, \\ 0 & \text{for } x \neq a. \end{cases}$$

Let $P_a := \Pi(\delta_a)$. From the relations

$$\delta_a^2 = \overline{\delta_a} = \delta_a, \qquad \delta_a\delta_b = 0 \quad \text{for } a \neq b, \qquad \sum_{a\in X} \delta_a = 1$$

we obtain

$$P_a^2 = P_a^* = P_a, \qquad P_a P_b = 0 \quad \text{for } a \neq b, \qquad \sum_{a\in X} P_a = 1.$$

The geometric meaning of these relations is the following.

The space V is a direct sum of orthogonal subspaces V_a, $a \in X$, and P_a is the orthoprojector on V_a.

The compatibility condition (25) applied to $g \in G$ and $\delta_a \in \mathcal{A}(X)$ gives:

$$\pi(g)P_a\pi(g)^{-1} = P_{a\cdot g^{-1}} \quad \text{or} \quad \pi(g)V_a \subset V_{a\cdot g^{-1}}.$$

Since $\pi(g)$ is a unitary operator, we conclude that all subspaces V_a have the same dimension, say k. Fix a k-dimensional Hilbert space W and somehow identify each V_a with W. An element $v \in V$ can be considered as a W-valued function f on X. Namely, the value $f(a)$ is equal to $P_a v \in V_a \simeq W$.

Finally, the compatibility condition implies that the value of the transformed function $\pi(g)f$ at the point $x \in X$ depends only on the value of the initial function f at $x\cdot g$. Therefore, π has the form (22), hence, according to Proposition 5, is induced from some representation ρ of the subgroup H. □

The representation-theoretic meaning of induced representations is revealed by

Proposition 6 (Frobenius Duality). Ind_H^G *defines a functor from* $\mathcal{R}ep\, H$ *to* $\mathcal{R}ep\, G$ *which is dual to the restriction functor* $\mathrm{Res}_H^G : \mathcal{R}ep\, G \to \mathcal{R}ep\, H$ *in the following sense:*

(27)

$$I(\pi, \mathrm{Ind}_H^G\rho) \cong I(\mathrm{Res}_H^G\pi, \rho) \quad \textit{for any } (\pi, V) \in \mathcal{R}ep\, G,\ (\rho, W) \in \mathcal{R}ep\, H.$$

Proof. We establish a one-to-one linear correspondence between the spaces $I(\pi, \operatorname{Ind}_H^G \rho)$ and $I(\operatorname{Res}_H^G \pi, \rho)$. Let $A: V \to L(X, W)$ be an intertwiner between (π, V) and $(\operatorname{Ind}_H^G \rho, L(X, W))$. Then for any $v \in V$ its image $f = Av$ is a W-valued function on X. Denote by $a(v)$ the value of this function at the initial point $x_0 = (H)$. We claim that the map $a: v \mapsto a(v)$ belongs to $\operatorname{Hom}_H(V, W) = i(\operatorname{Res}_H^G \pi, \rho)$. Indeed, $a(\pi(h)v) = (A\pi(h)v)(x_0) = (\operatorname{Ind}_H^G \rho\,(h)Av)(x_0) = \rho(h)Av(x_0) = \rho(h)a(v)$.

Conversely, if $a \in \operatorname{Hom}_H(V, W)$, we define the operator $A: V \to L(X, W)$ by $Av(x) = a(\pi(s(x))v)$. Then

$$\begin{aligned} A(\pi(g)v)(x) &= a(\pi(s(x)g)v) = a(\pi(h(x, g)s(x\cdot g))v) \\ &= \rho(h(x, g))a(\pi(s(x\cdot g))v) = \rho(h(x, g))Av(x\cdot g) \\ &= (\operatorname{Ind}_H^G(g)Av)(x). \end{aligned}$$

Therefore, A is an intertwiner between (π, V) and $\big(\operatorname{Ind}_H^G \rho, L(X, W)\big)$. From the definitions of A and a it is evident that the correspondence $a \longleftrightarrow A$ is a bijection. $\square$

Actually, the Frobenius duality can be used as an alternative definition of an induced representation. Indeed, let $\pi_1, \ldots, \pi_k$ be all (equivalence classes of) irreducible representations of G. Then any $\pi \in \mathcal{R}ep\, G$ is uniquely written as

$$\pi = m_1\pi_1 + \cdots + m_k\pi_k, \quad m_j \in \mathbb{Z}_+.$$

In particular, it is true for $\pi = \operatorname{Ind}_H^G \rho$. But from (27) we have

$$m_j = i\big(\operatorname{Ind}_H^G(\rho), \pi_j\big) = i\big(\rho, \operatorname{Res}_H^G(\pi_j)\big).$$

Therefore, all m_j are uniquely determined by the equivalence class of ρ. Hence, the equivalence class of π is determined by (23).

Three useful corollaries of this observation are:

Proposition 7 (Induction by stages). *If $K \subset H \subset G$ are two subgroups, then*

$$\operatorname{Ind}_H^G \operatorname{Ind}_K^H \simeq \operatorname{Ind}_K^G. \tag{28}$$

Proof. The proof follows from the evident relation $\operatorname{Res}_K^H \operatorname{Res}_H^G \simeq \operatorname{Res}_K^G$. $\square$

Proposition 8 (The structure of the regular representation). *Let $\rho := \operatorname{Ind}_{\{e\}}^G 1$ be the regular representation of G. Then*

$$\rho = \sum_{i=1}^k d_i \cdot \pi_i \quad \textit{where } d_i = \dim \pi_i.$$

In particular, we get the **Bernside formula** $\sum_{i=1}^k d_i^2 = |G|$.

Proposition 9 (Frobenius formula). *The character of the induced representation* $\pi = \operatorname{Ind}_H^G(\rho)$ *is given by the formula*

$$\chi_\pi(g) = \sum_{x \in G/H} \chi_\rho(xgx^{-1}) \tag{29}$$

where χ_ρ *is extended from* H *to* G *by zero.*

In conclusion we present the formula for the intertwining number for two induced representations.

Proposition 10 (Mackey Formula). *Let* H *and* K *be two subgroups of the finite group* G. *Let* (ρ, V) *and* (σ, W) *be representations of* H *and* K, *respectively. Then*

$$i\big(\operatorname{Ind}_H^G \rho,\ \operatorname{Ind}_K^G \sigma\big) = \dim L(G,\, H,\, K;\, \rho,\, \sigma) \tag{30}$$

where $L(G, H, K; \rho, \sigma)$ *is the space of* $\operatorname{Hom}(V, W)$*-valued operator functions* f *on* G *satisfying*

$$f(kgh) = \sigma(k) \circ f(g) \circ \rho(h). \tag{31}$$

□

In particular, for trivial representations ρ and σ we obtain

$$i\big(\operatorname{Ind}_H^G 1,\ \operatorname{Ind}_K^G 1\big) = |K \backslash G / H| \quad \text{(the number of double cosets).} \tag{32}$$

Note that the last number has three group-theoretic interpretations:

1) the number of K-orbits in $X = G/H$,

2) the number of H-orbits in $Y = K \backslash G$,

3) the number of G-orbits in $X \times Y$.

In many cases this simple formula already allows us to construct all irreducible representations of a given finite group G.

Example 3. Let G be the group of rotations of the solid cube. There are several homogeneous G-sets (the index shows the cardinality of the set):

- the set X_6 of faces;
- the set X_8 of vertices;
- the set X_{12} of edges;
- the set X_4 of big diagonals;
- the set X_2 of inscribed tetrahedrons;
- the set X_1 – the center of the cube.

Using the formula (32) we can fill up the table of intertwining numbers between the geometric representations $(\pi_i, L(X_i))$, $i = 1, 2, 4, 6, 8, 12$:

$i\backslash j$	1	2	4	6	8	12
1	1	1	1	1	1	1
2	1	2	1	1	2	1
4	1	1	2	1	2	2
6	1	1	1	3	2	3
8	1	2	2	2	4	4
12	1	1	2	3	4	7

Let us explain, for example, the equality $i(\pi_{12}, \pi_8) = 4$. The stabilizer of an edge is a subgroup K_2 of two elements in G. The set X_8 of eight vertices splits into four K_2-orbits. Another example: there are exactly two G-orbits in $X_6 \times X_8$; one consists of pairs (f, v) such that the face f contains the vertex v, another contains all other pairs. Therefore, $i(\pi_6, \pi_8) = 2$. The reader is advised to make an independent check of some other numbers in the table.

Now we show how to describe the set $\widehat{G}$ of all (equivalence classes of) irreducible representations of G by just contemplating the table above.

First, notice that π_1 is a trivial 1-dimensional representation. We include it in the set $\widehat{G}$ under the name ρ_1. We see from the table that it enters with multiplicity one in every π_i.

Further, since $i(\pi_2, \pi_2) = 2$, π_2 splits into two irreducible components: ρ_1 and another 1-dimensional representation ρ'_1, non-equivalent to ρ_1.

The second row of the table shows that this representation ρ'_1 enters in the decomposition of π_8 with multiplicity 1 and does not occur in π_4, π_6, or π_{12}.

From $i(\pi_4, \pi_4) = 2$ we conclude that π_4 is the sum of the "obligatory" ρ_1 and an irreducible 3-dimensional representation, which we denote ρ_3.

Since $i(\pi_6, \pi_6) = 3$, π_6 splits into three non-equivalent representations. The equality $i(\pi_6, \pi_4) = 2$ shows that one of them is ρ_1 and another is ρ_3. The remaining one we denote by ρ_2.

The equality $i(\pi_8, \pi_8) = 4$ means that π_8 is either a sum of two equivalent irreducible components, or splits into four pairwise non-equivalent irreducible components. The first possibility contradicts $i(\pi_8, \pi_1) = 1$. Furthermore, the equations $i(\pi_8, \pi_2) = i(\pi_8, \pi_4) = 2$ imply that π_8 contains ρ'_1 and ρ_3 with multiplicity 1. Together with the obligatory ρ_1 it gives the dimension 5. Therefore, the remaining component has dimension 3 and we denote it by ρ'_3.

The squares of dimensions of ρ_1, ρ'_1, ρ_2, ρ_3, ρ'_3 sum up to $24 = |G|$. According to the Bernside formula (Proposition 8), we have found all the representations of G.

We leave it to the reader to find the decomposition of π_{12}, to describe geometrically the intertwiners between $L(X_i)$ and $L(X_j)$, and to locate the irreducible components in $L(X_i)$. ♢

2.2. Induced representations of Lie groups.

Now we turn to the case of unitary representations (possibly infinite-dimensional) of Lie groups. Let G be a Lie group, let $H \subset G$ be a closed subgroup, and let $M = H\backslash G$ be a smooth manifold of right H-cosets in G. Let (ρ, W) be a unitary representation of H in a Hilbert space W. We want to define the induced representation $\operatorname{Ind}_H^G \rho$, which must be a unitary representation of G.

We cannot use the formulae (14), (16) because in general there is no G-invariant inner product in the space $L(G, H, \rho, W)$ even in the simplest case $(\rho, W) = (1, \mathbb{C})$ when $L(G, H, \rho, W)$ coincides with the function space $L(M)$.

However, in this case we can use the so-called **natural Hilbert space** $L^2(M)$ associated with the manifold M instead of $L(M)$. By definition, the space $L^2(M)$ consists of square-integrable sections of the line bundle L of half-densities on M. In a local coordinate system $(x^1, \ldots, x^n)$, $n = \dim M$, a section f of L has the form $f(x)\sqrt{|d^n x|}$ and under the change of coordinates $x = x(y)$ it transforms into $\tilde{f}(y)\sqrt{|d^n y|}$ where

$$\tilde{f}(y) = f(x(y)) \left| \det \left\| \frac{\partial x^i}{\partial y^j} \right\| \right|^{\frac{1}{2}}.$$

For two sections f_1 and f_2 the inner product is given by the integral

$$(f_1, f_2) = \int_M f_1 \overline{f_2}. \tag{33}$$

Here the expression $f_1\overline{f_2}$ is understood as a density on M, which in a coordinate chart U with coordinates $(x^1, \ldots, x^n)$ looks like $f_1(x)\overline{f_2(x)}|d^n x|$.

Since L is a natural bundle (see Appendix II.2.2), the group $G \subset \operatorname{Diff}(M)$ acts on L. If a point $m \in M$ has coordinates $(x^1, \ldots, x^n)$ in a chart U and the point $m \cdot g$ has coordinates $(y^1, \ldots, y^n)$ in a chart V, then y^j are functions of x^i and vice versa (for a fixed $g \in G$). Also a section $f = f(x)\sqrt{|d^n x|}$ goes to the section $\pi(g)f$, which has the form $\widetilde{f}(y)\sqrt{|d^n y|}$ in V where $\widetilde{f}(y) = f(x(y))|\det \|\partial x^j / \partial y^i\||^{\frac{1}{2}}$. From this definition it follows that $(\pi(g)f_1, \pi(g)f_2) = (f_1, f_2)$, so the representation $\big(\pi, L^2(M)\big)$ is unitary.

In practical computations one usually uses just one chart U with coordinates $(x^1, \ldots, x^n)$, which covers almost all of the manifold M except for a finite union of submanifolds of lower dimensions. We can also define a smooth section $s : U \to G$ of the natural projection $p : G \to M : g \mapsto Hg$. Then the big part of G (also except for a finite union of submanifolds of lower dimensions) will be identified with the product $H \times U$ via $g = hs(x)$, $h \in H$, $x \in U$.

Note that in general the fiber bundle $H \to G \xrightarrow{p} M$ is non-trivial, so that there is no smooth (or even continuous) section $s : M \to G$ of the projection p on the whole M.

Example 4. Let M be the two-dimensional sphere $S^2 \subset \mathbb{R}^3$. It is a homogeneous manifold $SO(2)\backslash SO(3)$. Geometrically, the group $SO(3)$ can be viewed as the tangent sphere bundle[3] T_1M over M. The absence of a continuous section $s : M \to G$ means that there is no continuous tangent vector field on S^2 that has unit length everywhere. This is the well-known "Hedgehog Theorem" from algebraic topology.

Nevertheless, we can cover all of the sphere except for one point by a coordinate chart U (see Appendix II.1.1) and identify $p^{-1}(U) \subset G$ with $U \times SO(2) \simeq \mathbb{R}^2 \times S^1$. ♢

Let $d_l^G(g)$ and $d_l^H(h)$ be left-invariant volume forms on G and H, respectively. Choose a local smooth section s over $U \subset M$ and identify $p^{-1}U \subset G$ with $H \times U$. In the parameters (h, x) a left-invariant form on G looks like

$$d_l^G(g) = r(h, x)\, d_l^H(h)\, d^n x$$

for some smooth function r on $U \times H$. Taking into account the left invariance of $d_l^G(g)$ and $d_l^H(h)$, we get $r(x, h'h) = r(x, h)$. Therefore, $r(x, h) = r(x)$. Let us define a special volume form μ_s on U by

$$\mu_s = r(x)\Delta_G(s(x))d^n x. \tag{34}$$

Then the following equality holds:

$$d_l^G(g) = \Delta_G(s(x))^{-1}\, d_l^H(h)\, d\mu_s(x). \tag{35}$$

Using the relations $d_r^G(g) = \Delta_G(g)d_l^G(g)$, $d_r^H(h) = \Delta_H(h)d_l^H(h)$, we can rewrite (35) in the form

$$d_r^G(g) = d_r^G\big(hs(x)\big) = \frac{\Delta_G(h)}{\Delta_H(h)} d_r^H(h) d\mu_s(x). \tag{35'}$$

The advantage of the measure μ_s is its simple transformation rule.

[3] That is, the collection of tangent vectors of unit length to M.

Lemma 3. *Under the action of G the measure μ_s is transformed according to the rule*

$$\frac{d\mu_s(x\cdot g)}{d\mu_s(x)}=\frac{\Delta_H(h_s(x,\,g))}{\Delta_G(h_s(x,\,g))} \tag{36}$$

where $h_s(x,\,g)\in H$ is defined by the Master equation

$$s(x)g=h_s(x,\,g)s(x\cdot g). \tag{37}$$

Proof. We shall, as above, use the pair $(h,\,x)\in H\times U$ to parametrize the element $hs(x)\in G$. Then the element $hs(x)g$ is parametrized by the pair $(h\cdot h_s(x,\,g),\,y)$ where $y=x\cdot g$ and $h_s(x,\,g)$ is defined by (37). Therefore for the right-invariant form d_r^G we have

$$d_r^G(h,\,x)=d_r^G(h\cdot h_s(x,\,g),\,x\cdot g).$$

From this, using (35′), we obtain

$$\frac{\Delta_G(h)}{\Delta_H(h)}d\mu_s(x)d_r^H(h)=\frac{\Delta_G(h\cdot h_s(x,\,g))}{\Delta_H\big(h\cdot h_s(x,\,g)\big)}d\mu_s(x\cdot g)d_r^H(h\cdot h(x,\,y)),$$

which is equivalent to the desired formula (36). □

Example 5. Let $G=SL(2,\,\mathbb{R})$ act from the right on the projective line $\mathbb{P}^1(\mathbb{R})$ with homogeneous coordinates $(x^0:x^1)$ by the formula

$$g=\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}:\ (x^0:x^1)\mapsto(x^0:x^1)\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}=(\alpha x^0+\gamma x^1:\beta x^0+\delta x^1).$$

Let U be the chart with coordinate $x=x^0/x^1$. It covers all of the projective line except for the point $(1:0)$. In terms of this coordinate the action of G is fractional linear: $x\mapsto\frac{\alpha x+\gamma}{\beta x+\delta}$.

The group G is unimodular (as any semisimple Lie group) and the bi-invariant volume form is

$$dg=\frac{d\alpha\wedge d\beta\wedge d\gamma}{\alpha}=-\frac{d\beta\wedge d\gamma\wedge d\delta}{\delta}=\frac{d\alpha\wedge d\beta\wedge d\delta}{\beta}=-\frac{d\alpha\wedge d\gamma\wedge d\delta}{\gamma}.$$

The subgroup H, the stabilizer of the point $x=0$, is the upper triangular subgroup with elements

$$h=\begin{pmatrix}\lambda&\mu\\ 0&\lambda^{-1}\end{pmatrix}.$$

This group is not unimodular and we have

$$d_l^H h = \frac{d\lambda \wedge d\mu}{\lambda^2}, \qquad d_r^H h = d\lambda \wedge d\mu, \qquad \Delta_H(h) = \lambda^2.$$

We choose the section $s:\ \mathbb{P}^1 \to G$

$$s(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

Then the Master equation (37) takes the form

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}.$$

The solution to this equation is

$$y = x \cdot g = \frac{\alpha x + \gamma}{\beta x + \delta}, \qquad \lambda = (\beta x + \delta)^{-1}, \qquad \mu = \beta.$$

The parametrization $g \mapsto (h,\, x)$ looks like

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \left(\begin{pmatrix} \delta^{-1} & \beta \\ 0 & \delta \end{pmatrix}, \frac{\gamma}{\delta} \right).$$

The equation (35′) takes the form

$$\left| \frac{d\beta \wedge d\gamma \wedge d\delta}{\delta} \right| = \left| \frac{d\delta^{-1} \wedge d\beta}{\delta^{-2}} \wedge d\mu_s\left(\frac{\gamma}{\delta}\right) \right|,$$

which implies $|d\mu_s(x)| = |dx|$.

The transformation law (36) in this case is

$$d\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) / dx = (\beta x + \delta)^{-2}.$$

The natural Hilbert space $L^2\big(\mathbb{P}^1(\mathbb{R})\big)$ consists of expressions $f(x)\sqrt{|dx|}$, $f \in L^2(\mathbb{R},\, dx)$, and the action of G in terms of functions $f(x)$ has the form

$$\left(\pi\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} f\right)(x) = |\beta x + \delta|^{-1} f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right).$$

This formula defines a unitary representation of $SL(2,\, \mathbb{R})$ in $L^2\big(\mathbb{P}^1(\mathbb{R})\big)$. $\diamondsuit$

The considerations above suggest the following general definition of the **unitary induced representation** $\operatorname{Ind}_H^G(\rho,\, W)$ for Lie groups.

The representation space $L^2(M, W) \simeq L^2(M) \boxtimes W$ consists of W-valued square-integrable half-densities on $M = H\backslash G$.

In a local chart U a half-density f can be written as $f(x)\sqrt{d\mu_s(x)}$, $f \in L^2(M, W, \mu_s)$. So, for a given section $s : U \to G$ the representation space is identified with $L^2(M, W, \mu_s)$.

The representation operators in this space are given by

$$(38) \qquad \big(\pi(g)f\big)(x) = \left(\frac{\Delta_H\big(h_s(x, g)\big)}{\Delta_G\big(h_s(x, g)\big)}\right)^{\frac{1}{2}} \rho\big(h_s(x, g)\big)f(x \cdot g).$$

This is the exact analog of formulas (16), (17) for finite groups. Note that the additional factor $\left(\frac{\Delta_H\left(h_s(x,g)\right)}{\Delta_G\left(h_s(x,g)\right)}\right)^{\frac{1}{2}}$ equals 1 for finite groups.

We can also give an alternative definition of the space $L^2(M, W)$ in terms of vector-functions on G, but it takes more preparation.

For any $f \in L^2(M, W, \mu_s)$ we define the W-valued function ϕ_f on G by

$$(39) \qquad \phi_f\big(hs(x)\big) = \left(\frac{\Delta_H(h)}{\Delta_G(h)}\right)^{\frac{1}{2}} \rho(h)f(x).$$

We denote the space of all ϕ_f, $f \in L^2(M, W, \mu_s)$, by $L^2(G, H, \rho, W)$. It is clear that all $\phi \in L^2(G, H, \rho, W)$ possess the property

$$(40) \qquad \phi(hg) = \left(\frac{\Delta_H(h)}{\Delta_G(h)}\right)^{\frac{1}{2}} \rho(h)\phi(g).$$

It turns out that this property is essentially characteristic for $L^2(G, H, \rho, W)$. To explain this we introduce a non-negative function $p(g)$ such that[4]

1) $\int_H p(hg)d_rh \equiv 1$ for all $g \in G$,
2) supp $p \cap Hg$ is compact for any coset Hg.

Lemma 4. *Assume that a W-valued strongly measurable function $\phi(g)$ satisfies* (40) *and also the finiteness condition*

$$(41) \qquad \int_G \|\phi(g)\|^2_W p(g)d_r(g) < \infty.$$

Then $\phi \in L^2(G, H, \rho, W)$.

[4]The existence of such a function is rather evident and the detailed construction is given in N. Bourbaki's book, *Integration*, Ch. VII, §2, no. 4.

Proof. Let $f(x) = \phi(s(x))$. If $f \in L^2(M, W, \mu_s)$, then $\phi = \phi_f$ and we are done. The function $\psi(g) = \|\phi(g)\|_W^2$ has the property $\psi(hs(x)) = \frac{\Delta_H(h)}{\Delta_G(h)}\psi(s(x)) = \frac{\Delta_H(h)}{\Delta_G(h)}\|f(x)\|_W^2$. Therefore, after the change of variables $g = hs(x)$, the integral (41) becomes

$$\begin{aligned}\int_G \psi(hs(x))p(hs(x))d_r^G(hs(x)) &= \int_{H\times M} \|f(x)\|_W^2 p(hs(x))d_r^H(h)d\mu_s(x)\\ &= \int_M \|f(x)\|_W^2 d\mu_s(x) = \|f\|^2_{L^2(M,\,W,\,\mu_s)}.\end{aligned}$$

We see that the finiteness condition (41) indeed is equivalent to finiteness of the norm $\|f\|_{L^2(M,\,W,\,\mu_s)}$. $\square$

Note that in the course of the proof of Lemma 4 we get the explicit formula for the inner product in $L^2(G, H, \rho, W)$:

$$(\phi_1, \phi_2) = \int_G (\phi_1(g), \phi_2(g))_W p(g) d_r^G(g) \tag{42}$$

where $p(g)$ is any function on G satisfying conditions 1) and 2) above.

Example 6. Keep the notation of Example 5. Let $\rho_{\sigma,\epsilon}$ be a 1-dimensional unitary representation of H defined by

$$\rho_{\sigma,\epsilon}\begin{pmatrix}\lambda & \mu\\ 0 & \lambda^{-1}\end{pmatrix} = |\lambda|^{i\sigma}(\operatorname{sign}\lambda)^\epsilon$$

where $\sigma \in \mathbb{R}$, $\epsilon = 0, 1$. Then the induced representation $\pi_{\sigma,\epsilon}$ acts in $L^2(\mathbb{R}, dx)$ by the formula

$$\left(\pi_{\sigma,\epsilon}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} f\right)(x) = |\beta x + \delta|^{-1+i\sigma}\big(\operatorname{sign}(\beta x + \delta)\big)^\epsilon f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right).$$

This is the so-called **principal series** of the unitary representation of G. $\diamondsuit$

2.3. $*$-representations of smooth G-manifolds. In this section we transfer the results of Section 1 to the case of G-manifolds. The most important results have to do with homogeneous manifolds.

Let M be a smooth manifold with an action of a Lie group G on it. We call M a **G-manifold**.

Definition 1. A **$*$-representation** Π of a smooth manifold M in a Hilbert space $\mathcal{H}$ is an algebra homomorphism $\Pi\colon \mathcal{A}(M) \to \operatorname{End}(\mathcal{H})$ satisfying the condition:

$$\Pi(\overline{\phi}) = \Pi(\phi)^*.$$

A $*$-representation Π is called **non-degenerate** if the set of vectors of the form $\Pi(\phi)x$, $\phi \in \mathcal{A}(M)$, $x \in \mathcal{H}$, is dense in $\mathcal{H}$.

Definition 2. We say that a $*$-representation Π of a smooth G-manifold M in a Hilbert space $\mathcal{H}$ is **compatible** with a unitary representation $(\pi, \mathcal{H})$ of the group G in the same space if the following condition is satisfied:

$$\pi(g) \circ \Pi(\phi) \circ \pi(g)^{-1} = \Pi\big(R(g)\phi\big) \quad \text{where } \big(R(g)\phi\big)(m) = \phi(m \cdot g).$$

Let us discuss these definitions and draw some consequences from them.

First of all, observe that the non-degeneracy condition in Definition 1 for compact manifolds is equivalent to the equality $\Pi(1_M) = 1_{\mathcal{H}}$. It is introduced mainly to avoid the trivial example $\Pi(\phi) \equiv 0$.

Second, instead of operators $\Pi(\phi)$ we can define the **projector-valued measure** μ on M so that the operators $\Pi(\phi)$ are given by integrals:

$$\Pi(\phi) = \int_M \phi(m) d\mu(m). \tag{43}$$

For the reader's convenience we recall the main facts here about projector-valued measures and the corresponding integrals (see, e.g., [**KG**] or any textbook on spectral theory of operators in Hilbert spaces).

Let X be a topological space. Denote by $\mathcal{B}(X)$ the minimal collection of subsets of X that contains all open subsets and is closed under countable unions and complements. Elements of $\mathcal{B}(X)$ are called **Borel subsets**.

A real-valued function f is called a **Borel function** if for all $c \in \mathbb{R}$ the sets

$$L_c(f) := \{x \in X \mid f(x) \leq c\}$$

are Borel sets. It is known that the set of all Borel functions is the minimal algebra that is closed under pointwise limits and contains all continuous functions. In other words, any Borel function can be obtained from continuous functions by a sequence of pointwise limits.

Warning. Note that one limit is not enough! Actually, the situation here is rather delicate. The functions that can be written as a pointwise limit of a sequence of continuous functions form the so-called 1st Baire class. In a complete metric space the function of this class always has a dense subset of points of continuity.

The pointwise limits of sequences of functions from the 1st Baire class form the 2nd Baire class. The functions of this class can be nowhere continuous but still do not exaust the set of Borel functions.

We can inductively define the k-th Baire class for all $k \in \mathbb{N}$ and call the union of these classes the ω-th Baire class. It still does not exaust the set of Borel functions.

So, we can define new Baire classes: $\omega+1$-st, $\omega+2$-nd, ... , 2ω-th, ... , 3ω-th, ... , ω^2-th, ... , ω^ω-th, etc.

The collection of all Baire classes is a minimal uncountable set: any Baire class contains only a countable family of smaller Baire classes. ♠

A complex-valued function is called Borel if its real and imaginary parts are Borel functions.

We denote by $\mathcal{P}(\mathcal{H})$ the set of all **orthoprojectors** in a Hilbert space $\mathcal{H}$, i.e. operators P satisfying $P^2 = P = P^*$.

By definition a projector-valued measure μ on X is a map from $\mathcal{B}(X)$ to $\mathcal{P}(\mathcal{H})$ that satisfies the conditions:

1) $\mu(X) = 1_{\mathcal{H}}$.

2) $\mu\left(\bigsqcup_{i=1}^\infty S_i\right) = \sum_{i=1}^\infty \mu(S_i)$ where $\bigsqcup$ denotes the union of disjoint sets.

The last property is usually called the **countable additivity**.

Remark 5. The additivity property is a very strong restriction on a projector-valued function. The point is that a sum $P_1 + P_2$ of two orthoprojectors can be an orthoprojector only if $P_1P_2 = P_2P_1 = 0$.[5] Geometrically this means that subspaces $\mathcal{H}_1 = P_1\mathcal{H}$ and $\mathcal{H}_2 = P_2\mathcal{H}$ are orthogonal.

Therefore, the additivity of μ implies that for any decomposition of a set S into a union of disjoint sets $S_1, S_2, \ldots$ we have a decomposition of the Hilbert space $V = \mu(S)\mathcal{H}$ into the orthogonal sum of subspaces $V_k = \mu(X_k)H$, $i = 1, 2, \ldots$. ♡

For a scalar function f on X we can define the integral of f with respect to a $\mathcal{P}(\mathcal{H})$-valued measure μ, which is denoted by $A = \int_X f(x)d\mu(x)$.

By definition it is an operator A in $\mathcal{H}$ such that for any two vectors $v_1, v_2 \in \mathcal{H}$ we have

$$(Av_1,\, v_2) = \int_X f(x)d\mu_{v_1,\, v_2}(x)$$

where the complex measure $\mu_{v_1,\,v_2}$ is defined by $\mu_{v_1,\,v_2}(S) = (\mu(S)v_1,\, v_2)$.

One can show that the integral exists for any bounded Borel function f. If f is continuous and has a compact support, then the integral can be defined as a uniform limit of the Riemann integral sums

$$S(\{V_i\},\, \{x_i\};\, f) = \sum_i f(x_i)\mu(V_i)$$

where $\{V_i\}$ is a covering of X by small Borel subsets, $x_i \in V_i$ are arbitrary chosen representatives, and the limit is taken over any sequence of coverings with shrinking elements.

[5] Check this using the algebraic definition of orthoprojector: $P^2 = P^* = P$.

For any $*$-representation $(\Pi, \mathcal{H})$ of a smooth manifold M we can associate a projector-valued measure μ on M as follows. For an open subset $O \in M$ we can define $\mu(O)$ as the orthoprojector on the closed subspace in $\mathcal{H}$ spanned by vectors of the form $\Pi(\phi)v$, $\phi \in \mathcal{A}(O)$, $v \in \mathcal{H}$.

For more general Borel sets the measure μ can be uniquely defined using the property of countable additivity.

Conversely, for any projector-valued measure μ on M we define a $*$-representation Π of M by (38).

In terms of the projector-valued measure μ the compatibility condition takes the form

$$\pi(g) \circ \mu(S) \circ \pi(g)^{-1} = \mu(S \cdot g^{-1}) \quad \text{for any Borel subset } S \subset M.$$

It is possible to describe all representations of a smooth manifold M. Comparing this situation with the one described in Section 1, we can suggest the following standard model for Π.

Let W be a Hilbert space, and let ρ be any Borel measure on M. Define the new Hilbert space $V = L^2(M, W, d\rho)$ as the space of square-integrable W-valued strongly measurable functions on M with the inner product

$$(f_1, f_2)_V = \int_M (f_1(x), f_2(x))_W d\rho. \tag{44}$$

Then we define the operator $\Pi(\phi)$ in V by the formula

$$(\Pi(\phi)f)(x) = \phi(x)f(x). \tag{45}$$

The projector-valued measure μ in this case is defined by

$$(\mu(S)f)(x) = \begin{cases} f(x) & \text{when } x \in S, \\ 0 & \text{otherwise.} \end{cases} \tag{46}$$

In other words, the operator $\mu(S)$ is the multiplication by the characteristic function of the set S.

It is rather clear that the equivalence class of the $*$-representation (Π, V) thus obtained depends only on the equivalence class of the measure μ and the Hilbert dimension k of W. So, we denote it by $\left(\Pi_{\mu,k}, V_{\mu,k}\right)$ or simply $\Pi_{\mu,k}$.

It turns out that this is not a universal model. To obtain the most general type of representation, we have to make the space W dependent on the point $m \in M$. This can be done and the final construction looks as follows.

Proposition 11. *Any non-degenerate $*$-representation Π of M is (equivalent to) a direct sum of representations $\Pi_{\mu_k,k}$, $k = 1, 2, \ldots, \infty$.*

The measures $\{\mu_k\}$ are pairwise disjoint and defined by the representation Π uniquely up to equivalence. □

Note that for $M = \mathbb{R}^1$ (resp. for $M = S^1$) this is essentially the Spectral Theorem for a self-adjoint (resp. unitary) operator in a Hilbert space.

Fortunately, the initial model (with a constant W) is still universal in the case of representations of homogeneous manifolds compatible with a given group representation. (Compare again with Section 1.)

Theorem 7. *Let $M = H\backslash G$ be a homogeneous right G-manifold, and let μ be a Borel measure on M given by some non-vanishing smooth density ρ. Assume that $(\Pi, \mathcal{H})$ is a $*$-representation of M compatible with a unitary representation $(\pi, \mathcal{H})$ of the group G.*

Then the Hilbert space $\mathcal{H}$ has the form $\mathcal{H} = L^2(M, W, d\mu)$ for some Hilbert space W, the representation Π is given by (40), and π has the form

$$(\pi(g)f)(x) = A(x, g)f(x \cdot g)\sqrt{\frac{d\mu(x \cdot g)}{d\mu(x)}} \tag{47}$$

where A is a unitary operator-valued function on $M \times G$ satisfying the **cocycle equation***:*

(48)

$$A(x, g_1)A(x \cdot g_1, g_2) = A(x, g_1 g_2) \quad \textit{for almost all } x, g_1, g_2 \in M \times G \times G.$$

Proof. For any $g \in G$ the operator $R(g)$ defines an automorphism of the algebra $\mathcal{A}(M)$. The compatibility condition implies that the $*$-representations Π and $\widetilde{\Pi} := \Pi \circ R(g)$ are equivalent. It follows (see Proposition 11) that all measures μ_k are quasi-invariant under the action of G on M. We now use the following fact from measure theory.[6]

Proposition 12. *All non-zero Borel measures on the homogeneous manifold $M = H\backslash G$ that are quasi-invariant under the action of G are pairwise equivalent. In particular, they are equivalent to the measure μ from Theorem 7 and to all measures μ_s introduced by formula (33) above.* □

In the case under consideration we conclude that from the measures μ_k, $k = 1, 2, \ldots, \infty$, only one measure, say μ_n, is non-zero and equivalent to the measure μ.

[6] This result is a generalization of Lemma 2 in Chapter 2 and can be proven in a similar way.

Fix a Hilbert space W of dimension n. The representation space $\mathcal{H}$ can be identified with $L^2(M, W, \mu)$ so that operators $\Pi(\phi)$, $\phi \in \mathcal{A}(M)$, act according to formula (47).

Now compare the representation operator $\pi(g)$ with the unitary operator

$$\big(\widetilde{\pi}(g)f\big)(x) = \sqrt{\frac{d\mu(x \cdot g)}{d\mu(x)}}\, f(x \cdot g).$$

It is clear that these two operators have the same commutation relations with operators $\Pi(\phi)$, namely

$$(25') \qquad \widetilde{\pi}(g)\Pi(\phi)\widetilde{\pi}(g)^{-1} = \Pi\big(R(g)\phi\big) \quad \text{where } \big(R(g)\phi\big)(x) = \phi(x \cdot g).$$

It follows that the operator $\pi(g)\widetilde{\pi}(g)^{-1}$ commutes with all multiplication operators $\Pi(\phi)$, $\phi \in \mathcal{A}(M)$. Therefore, this operator itself is multiplication by an operator-valued function $A(x, g)$. (We leave it to the reader to prove this statement using the scheme followed in Chapter 2.) So, our representation acquires the desired form (47).

The cocycle equation for $A(x, g)$ has the same form as in (23) for finite groups. It is deduced in the same way from the equality $\pi(g_1g_2) = \pi(g_1)\pi(g_2)$, but now we can only assert that for any $g_1, g_2 \in G$ the equation is satisfied for almost all $m \in M$. □

2.4. Mackey Inducibility Criterion.

The main result of this section is the criterion for when a representation of a Lie group G is induced from a representation of a closed subgroup $H \subset G$. It has the same form as in the finite group case (see Section 1), but the proof is much more involved.

Theorem 8 (Mackey Inducibility Criterion). *Let G be a Lie group, let $H \subset G$ be a closed subgroup, and let $M = H\backslash G$ be the smooth manifold of right H-cosets in G. Assume that the unitary representation (π, V) of G is compatible with a non-degenerate $*$-representation (Π, V) of the algebra $\mathcal{A}(M)$. Then there exists a unitary representation (ρ, W) of H such that π is equivalent to the induced representation $\operatorname{Ind}_H^G \rho$.*

Actually, Mackey proved a more general theorem (instead of smooth homogeneous manifolds he considered arbitrary homogeneous spaces for locally compact groups).

Proof. According to Theorem 7, we can write π in the form (47). So, we only need the following result.

Proposition 13. *Any solution to the cocycle equation* (48) *has the form*

$$A_{C,s,\rho}(m,\,g) = C(m)^{-1}\rho(h_s(m,\,g))C(m\cdot g) \tag{49}$$

where C is a unitary operator-valued function on M, ρ is a unitary representation of H in W, $s : M \to G$ is a section of the natural projection $p : G \to M$, and $h_s(m,\,g)$ is the solution to the master equation (37).

Proof. The proof of the statement for finite groups was given in Section 1. Here we shall follow the same scheme.

Technically, it is convenient to consider instead of the operator-valued function $A(x,\,g)$ on $M\times G$ another operator-valued function B on $M\times H\times M$ given by

$$B(x,\,h,\,y) = A\big(x,\,s(x)^{-1}hs(y)\big) \tag{50}$$

where $s : M \to G$ is a section of the natural projection $p : G \to M : g \mapsto Hg$.

The cocycle equation in terms of B acquires a more simple form

(51)

$B(x,\,h_1,\,y)B(y,\,h_2,\,z) = B(x,\,h_1h_2,\,z)$ for almost all values of arguments.

We can interpret $B(x,\,h,\,y)$ as a "transfer operator" acting from some space W_y attached to the point $y \in M$ to the space W_x attached to the point $x \in M$ and dependent on the auxiliary parameter $h \in H$. Equation (51) is a compatibility condition: the transfer in two steps with parameters h_1, h_2 is the same as the direct transfer with the parameter h_1h_2.

Proposition 13 can be rewritten as follows:

Proposition 13′. *The general solution to* (51) *has almost everywhere the form*

$$B(x,\,h,\,y) = C(x)^{-1}\rho(h)\,C(y) \tag{52}$$

where C is a unitary operator-valued function on M and ρ is a unitary representation of H in W.

It is clear that Proposition 13′ implies the Inducibility Criterion: after the substitution $f(x) \mapsto C(x)^{-1}f(x)$ the factor $A(x,\,g)$ in (47) takes the desired form $A(x,\,g) = \rho\big(h_s(x,\,g)\big)$.

Proof of Proposition 13′. We can delete from M finitely many submanifolds of smaller dimension, so that the rest can be covered by one chart U and choose s smooth everywhere on U.

The main difficulty is that the main relation (51) is valid only almost everywhere. To get a true relation we have to integrate it against some test functions.

So, let α, β be some W-valued functions on U, and let γ, ξ, η be some scalar functions on U, H, H, respectively. Also, we introduce the following notation:

$$\begin{aligned} f_{\alpha,\xi}(x) &= \int_{U\times H} B(x,\, h,\, y)\alpha(y)d\mu_s(y)\xi(h)d_l^H(h), \\ c_{\alpha,\xi,\beta} &= \int_{U\times H\times U} \big(B(x,\, h,\, y)\alpha(y),\, \beta(x)\big)_W \xi(h)d\mu_s(x)d_l^H(h)d\mu_s(y). \end{aligned}$$

Applying (51) to $\alpha(y)$, taking the inner product with $\beta(x)$, and using the property $B(x,\, h,\, y)^* = B(y,\, h^{-1},\, x)$, we get

$$\int_U \big(f_{\alpha,\xi}(x),\, f_{\beta,\eta}(x)\big)_W \gamma(x)dx = \int_U c_{\alpha,\,\xi*\eta,\,\beta}\gamma(x)dx$$

or

$$(53) \qquad \big(f_{\alpha,\xi}(x),\, f_{\beta,\eta}(x)\big)_W = c_{\alpha,\,\xi*\eta,\,\beta}.$$

Note that the right-hand side in (53) does not depend on $x \in U$. Geometrically, it means that the systems of vectors $\{f_{\alpha,\xi}(x)\}$ for different x's are congruent. In other words, there exist a system of vectors $\{v_{\alpha,\xi}\}$ in W and for any $x \in U$ a unitary operator $C(x)$ in W such that

$$(54) \qquad f_{\alpha,\xi}(x) = C(x)v_{\alpha,\xi}.$$

Let us replace the cocycle $B(x,\, h,\, y)$ by an equivalent cocycle

$$\widetilde{B}(x,\, h,\, y) = C(x)B(x,\, h,\, y)C^*(y)$$

and modify accordingly the definitions of $f_{\alpha,\xi}$ and $c_{\alpha,\,\xi*\eta,\,\beta}$. Then equation (54) becomes

$$\widetilde{f}_{\alpha,\xi}(x) \equiv \widetilde{v}_{\alpha,\xi}.$$

It follows that $\widetilde{B}(x,\, h,\, y)$ actually does not depend on x. Since $B(x,\, h,\, y)^* = B(y,\, h^{-1},\, x)$, it also does not depend on y. Finally, by (53) $\widetilde{B}(x,\, h,\, y) = \rho(h)$ for some unitary representation $(\rho,\, W)$ of the subgroup H. □

We shall use the Mackey Criterion to show that all unirreps of exponential Lie groups and many unirreps of more general Lie groups are induced from representations of smaller subgroups. To this end we need some additional information about $*$-representations of homogeneous manifolds.

Let M be a G-manifold. We say that a partition of M into G-orbits is **tame** if there exists a countable family $\{X_i\}_{i\in\mathbb{N}}$ of G-invariant Borel subsets in M which **separates** the orbits. (This means that for any two different orbits we can find an index i such that X_i contains one of these orbits but not the other.) For brevity we shall simply say that M is a **tame G-manifold**.

The following result plays a crucial role.

Theorem 9. *Let G be a Lie group, let M be a tame G-manifold, and let $(\Pi, \mathcal{H})$ be a $*$-representation of M compatible with an irreducible unitary representation $(\pi, \mathcal{H})$ of G. Then*

a) *The corresponding projector-valued measure μ on M is concentrated on a single G-orbit $\Omega \in M$.*

b) *The unirrep π is induced from a unirrep ρ of the stabilizer H of a point in Ω.*

Proof. a) For any G-invariant subset $X \in M$ the orthoprojector $\mu(X)$ commutes with $\pi(g)$, $g \in G$. Therefore the range of $\mu(X)$ is a G-invariant closed subspace in $\mathcal{H}$. But π is irreducible, hence the only such subspaces are $\{0\}$ and $\mathcal{H}$. We conclude that $\mu(X)$ is either the identity or the zero operator.

Now consider the family $\{X_i\}_{i\in\mathbb{N}}$ of G-invariant Borel subsets in M that separates the orbits. Replacing, if necessary, X_i by $M\backslash X_i$ we can assume that $\mu(X_i) = 0$ for all $i \in \mathbb{N}$. Then $\mu(\bigcup_i X_i) = 0$. Let $\Omega = M\backslash \bigcup_i X_i$. We claim that Ω is a single G-orbit. Indeed, suppose that Ω contains two different orbits Ω_1 and Ω_2. By the assumption, there is an X_k that separates these orbits, say $\Omega_1 \subset X_k$, $\Omega_2 \not\subset X_k$. But this contradicts the definition of Ω. By the construction we have $\mu(\Omega) = 1$.

b) The set Ω, being a G-orbit, has the structure of a smooth homogeneous G-manifold $\Omega = G/H$ where H is a stabilizer of a point in Ω. Moreover, since M is a tame G-manifold, Ω is a locally closed subset in M, hence a smooth submanifold.

According to the Inducibility Criterion the unirrep π is induced from some representation ρ of H. This representation is necessarily irreducible, since if $\rho = \rho_1 \oplus \rho_2$, then we would have $\pi = \mathrm{Ind}_H^G \rho_1 \oplus \mathrm{Ind}_H^G \rho_2$, which contradicts the assumption that π is irreducible. $\square$

We now use Theorem 9 and the Mackey Inducibility Criterion to prove the following statement.

Theorem 10. *Any unitary irreducible representation of an exponential Lie group G is induced by a 1-dimensional unitary representation of a closed subgroup $H \subset G$.*

Proof. We start by listing some properties of exponential Lie groups.

It is known that the class of exponential Lie groups is strictly between nilpotent and solvable simply connected Lie groups. A useful criterion is given by

Proposition 14. *Let G be a simply connected Lie group with* $\mathrm{Lie}\,(G) = \mathfrak{g}$. *Then G is exponential iff for any $X \in \mathfrak{g}$ the operator* $\mathrm{ad}\,X$ *has no non-zero pure imaginary eigenvalues.* □

It is also known that any non-abelian exponential Lie group G contains a non-central connected abelian normal subgroup A. We use this subgroup to construct a G-manifold M and a $*$-representation Π of M compatible with a given irreducible representation $(\pi, \mathcal{H})$ of G.

Since G is exponential, the subgroup A has the form $A = \exp \mathfrak{a}$, $\mathfrak{a} = \mathrm{Lie}\,(A)$. Hence, $A \approx \mathbb{R}^p$ with coordinates $x^1, \ldots, x^p$. Let $\widehat{A}$ be the Pontryagin dual to A, i.e. the group of all characters (1-dimensional unirreps) of A. These characters are labelled by points of the dual vector space $(\mathbb{R}^p)^*$ with coordinates $\lambda_1, \ldots, \lambda_p$ and have the form

$$\chi_\lambda(x) = e^{2\pi i \langle \lambda, x\rangle} \quad \text{where } \langle \lambda, x\rangle = \sum_k \lambda_k x^k.$$

Now let $(\pi, \mathcal{H})$ be a unitary representation of G. Put $M = \widehat{A}$ and define the $*$-representation $(\Pi, \mathcal{H})$ of M by the formula

$$\Pi(\phi) = \int_A \widetilde{\phi}(x)\pi(x)d^p x \tag{55}$$

where

$$\widetilde{\phi}(x) := \int_{\widehat{A}} \phi(\lambda)\chi_\lambda(x)d^p\lambda$$

is the Fourier transform of ϕ. That this is indeed a representation of $\mathcal{A}(M)$ follows from the known property of the Fourier transform:

$$(\phi_1 \cdot \phi_2)^{\sim} = \widetilde{\phi_1} * \widetilde{\phi_2}.$$

Moreover, this representation is non-degenerate, since for an appropriate $\phi \in \mathcal{A}(\widehat{A})$ the function $\widetilde{\phi}$ is almost concentrated in a given neighborhood of $0 \in A$, so that $\Pi(\phi)$ is close to 1 in the strong operator topology.

Finally, the representation (55) is compatible with π if we define the G-action on $\widehat{A}$ in a natural way:

$$\chi_{g\cdot\lambda}(x) = \chi_\lambda\big(g^{-1}xg\big), \qquad x \in A,\ \lambda \in \widehat{A}. \tag{56}$$

From Proposition 14 we obtain that the action of G on $\widehat{A}$ is a linear action given by matrices without pure imaginary eigenvalues. It follows that this action is tame.

Therefore, if π is irreducible, then, according to Theorem 9, π is induced from some representation of a stabilizer of a point in $\lambda \in \widehat{A}$. Since A is a non-central normal subgroup, the action of G is non-trivial and most points have stabilizers that are proper subgroups of G. In this case we can use induction on $\dim G$ to prove our statement.

As for exceptional points $\lambda \in \widehat{A}$ that are fixed by the whole group G, they correspond to degenerate representations of G, which are actually representations of some quotient group. So, we can again use induction on $\dim G$. (We omit the details since they are discussed in the main part of the book.) $\square$

References

[AW] S. Agnihotri and C. Woodward, *Eigenvalues of products of unitary matrices and quantum Schubert calculus*, Math. Res. Lett. **5** (1998), 817–836.

[AS] A. Alexeev and S. Shatashvili, *Path integral quantization of the coadjoint orbit of the Virasoro group and 2d gravity*, Nuclear Phys. **B323** (1989), 719–733; *From geometric quantization to conformal field theory*, Commun. Math. Phys. **128** (1990), 197–212.

[Ar1] V. I. Arnold, *Snake calculus and the combinatorics of the Bernoulli, Euler and Springer numbers of Coxeter groups*, Uspekhi Mat. Nauk **47** (1993), no. 1, 3–45; English transl., Russian Math. Surveys **47** (1992), 1–51.

[Ar2] ———, *On the teaching of mathematics*, Uspekhi Mat. Nauk **53** (1998), no. 1, 229–234; English transl., Russian Math. Surveys **53** (1998), 229–236.

[AG] V. I. Arnold and A. B. Givental, *Symplectic geometry*, Encyclopaedia of Mathematical Sciences, vol. 4, Dynamical systems. IV, Springer-Verlag, Berlin–Heidelberg–New York, 1990, pp. 1–136.

[A] M. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. **14** (1982), 1–15.

[AK] L. Auslander and B. Kostant, *Polarization and unitary representations of solvable Lie groups*, Invent. Math. **14** (1971), 255–354.

[Bar] D. Barbasch, *The unitary dual for complex classical Lie groups*, Invent. Math. **96** (1989), 103–176.

[BV] D. Barbash and D. Vogan, *Unipotent representations of complex semisimple groups*, Ann. of Math. (2) **121** (1985), 41–110.

[Ba] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. of Math. (2) **48** (1947), 568–640.

[BPZ] A. Belavin, A. Polyakov, and A. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nuclear Phys. **B241** (1984), 333–380.

[Be1] F. A. Berezin, *Some remarks on the associative envelope of a Lie algebra*, Funkcional. Anal. i Prilozhen. **1** (1967), no. 2, 1–14; English transl. in Functional Anal. Appl. **1** (1967).

[Be2] ______, *Introduction to algebra and analysis with anticommuting variables*, Izd. Moskow. Univ., Moscow, 1983; English transl., *Introduction to supersymmetric analysis*, Reidel, Dordrecht.

[BK] F. A. Berezin and G. I. Kac, *Lie groups with commuting and anticommuting parameters*, Mat. Sb. **82** (1970), 343–359; English transl. in Math. USSR-Sb.

[BGV] N. Berlin, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Springer-Verlag, Berlin–Hiedelberg–New York, 1992.

[BCD] P. Bernat, N. Conze, M. Duflo, M. Levy-Nahas, M. Rais, P. Renouard, and M. Vergne, *Representations des groupes de Lie resolubles*, Dunod, Paris, 1972.

[BS] R. Berndt and R. Schmidt, *Elements of the representation theory of the Jacobi group*, Birkhäuser, Basel, 1998.

[BL] I. N. Bernstein and D. A. Leites, *Irreducible representations of finite dimensional Lie superalgebras of series W and S*, C. R. Acad. Bulgare Sci. **32** (1979), 277–278.

[B] R. J. Blattner, *On a theorem of G. W. Mackey*, Bull. Amer. Math. Soc. **68** (1962), 585; *Positive definite measures*, Proc. Amer. Math. Soc. **14** (1963), 423–428.

[Br] I. D. Brown, *Dual topology of a nilpotent Lie group*, Ann. Sci. École Norm. Sup. **6** (1973), 407–411.

[Bor] A. Borel, *Linear algebraic groups*, Springer-Verlag, New York, 1991.

[BW] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups and representations of reductive groups*, Princeton Univ. Press, Princeton, NJ, 1980.

[Bou] N. Bourbaki, *Lie groups and Lie algebras, Chapters 1–3*, Springer-Verlag, Berlin–Hiedelberg–New York, 1989, 1998; *Chapters 4–6*, Springer, Berlin–Hiedelberg–New York, 2002.

[Bu] I. K. Busyatskaya, *Representations of exponential Lie groups*, Funkcional. Anal. i Prilozhen. **7** (1973), no. 2, 151–152; English transl. in Functional Anal. Appl. **7** (1972).

[Di1] J. Dixmier, *Sur les représentation unitaires des groupes de Lie nilpotents,* I, Amer. J. Math. **81** (1959), 160–170; II, Bull. Soc. Math. France **85** (1957), 325–388; III, Canad. J. Math. **10** (1958), 321–348; IV, Canad. J. Math. **11** (1959), 321–344; V, Bull. Soc. Math. France **87** (1959), 65–79; VI, Canad. J. Math. **12** (1960), 324–352.

[Di2] ______, *Enveloping algebras*, North Holland, Amsterdam, 1977.

[Di3] ______, *C^*-algebras*, North-Holland, Amsterdam, 1977.

[DW] A. H. Dooley and N. J. Wildberger, *Harmonic analysis and the global exponential map for compact Lie groups*, Canad. Math. Bull. **27** (1993), 25–32; N. J. Wildberger, *On a relationship between adjoint orbits and conjugacy classes of a Lie group*, Canad. J. Math. **33** (1990), no. 3, 297–304.

[Dr] V. G. Drinfeld, *Quantum groups*, Proc. Internat. Congress Math. (Berkeley, CA, 1986), Amer. Math. Soc., Providence, RI, 1987, pp. 798–820.

[DKN] B. A. Dubrovin, I. M. Krichever, and S. P. Novikov, *Integrable systems. I*, Encyclopaedia of Mathematical Sciences, Dynamical systems. IV, vol. 4, Springer-Verlag, Berlin–Hiedelberg–New York, 1990, pp. 173–280.

[Du1] M. Duflo, *Fundamental-series representations of a semisimple Lie group*, Funkcional. Anal. i Prilozhen. **4** (1970), no. 2, 38–42; English transl. in Functional Anal. Appl. **4** (1970).

[Du2] ———, *Représentations irréductibles des groupes semi-simples complexes*, Analyse harmonique sur les groupes de Lie (Sém. Nancy–Strasbourg, 1973–75), Lecture Notes in Math., vol. 497, Springer-Verlag, Berlin–Hiedelberg–New York, 1975, pp. 26–88.

[Du3] ———, *Operateurs differentiels bi-invariants sur un groupe de Lie*, Ann. Sci. École Norm. Sup. **4** (1977), 265–283.

[EFK] P. Etingof, I. Frenkel, and A. Kirillov, *Lectures on representation theory and Knizhnik-Zamolodchikov equations*, Amer. Math. Soc., Providence, RI, 1998.

[EK] P. Etingof and A. Kirillov, Jr., *The unified representation-theoretic approach to special functions*, Funkcional. Anal. i Prilozhen. **28** (1994), no. 1, 91–94; English transl. in Functional Anal. Appl. **28** (1994).

[FF1] B. L. Feigin and D. B. Fuchs, *On invariant differential operators on the line*, Funkcional. Anal. i Prilozhen. **13** (1979), no. 4, 91–92; English transl. in Functional Anal. Appl. **13** (1979); *Homology of the Lie algebra of vector fields on the line*, Funkcional. Anal. i Prilozhen. **14** (1980), no. 3, 45–60; English transl. in Functional Anal. Appl. **14** (1980).

[FF2] ———, *Representations of the Virasoro algebra*, Advanced studies in contemporary mathematics (A. M. Vershik and D. P. Zhelobenko, eds.), Representation of Lie groups and related topics, vol. 7, Gordon and Breach, London, 1990.

[Fe] J. M. G. Fell, *Weak containment and induced representations of groups*, Canad. J. Math. **14** (1962), 237–268.

[Fr] I. Frenkel, *Orbital theory for affine Lie algebras*, Invent. Math. (1984), 301–352.

[FH] W. Fulton and J. Harris, *Representation theory. A first course*, Springer-Verlag, Berlin–Hiedelberg–New York, 1991.

[Ge] I. M. Gelfand, *Collected papers*, vol. II, Springer-Verlag, Berlin–Hiedelberg–New York, 1988.

[GK] I. M. Gelfand and A. A. Kirillov, *Sur les corps liés aux algèbres enveloppantes des algèbres de Lie*, Publ. Math. IHES (1966), no. 31, 509–523.

[Gi] V. A. Ginzburg, *The method of orbits in the representation theory of complex Lie groups*, Funkcional. Anal. i Prilozhen. **15** (1981), no. 1, 23–37; English transl. in Functional Anal. Appl. **15** (1981); *$\mathfrak{g}$-modules, Springer's representations and bivariant Chern classes*, Adv. Math. **61** (1986), 1–48.

[GO] P. Goddard and D. Olive (eds.), *Kac-Moody and Virasoro algebras. A reprint volume for physicists*, World Scientific, Singapore, 1988.

[GWW] C. Gordon, D. Webb, and S. Wolpert, *One can't hear the shape of a drum*, Bull. Amer. Math. Soc. **27** (1992), no. 1, 134–138.

[GW] C. Gordon and S. Wilson, *The spectrum of the Laplacian on Riemannian Heisenberg manifolds*, Michigan Math. J. **33** (1986), 253–271; *Isospectral deformations of compact solvmanifolds*, J. Differential Geom. **19** (1984), 241–256.

[GG] M. Goto and F. Grosshans, *Semisimple Lie algebras*, Marcel Dekker, New York–Basel, 1978.

[Gr] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. **9** (1957), 119–221.

[Grz] P. Ya. Grozman, *Local invariant bilinear operators on tensor fields on the plane*, Vestnik Mosk. Univ. Ser. I, Mat. Mekh. **6** (1980), 3–6; *Classification of bilinear invariant operators on tensor fields*, Funkcional. Anal. i Prilozhen. **14**, no. 2, 58–59; English transl. in Functional Anal. Appl. **14** (1980).

[Gu] A. Guichardet, *Théorie de Mackey et méthode des orbites selon M. Duflo*, Exposition. Math. **3** (1985), 303–346.

[GLS] V. Guillemin, E. Lerman, and S. Sternberg, *Symplectic fibrations and multiplicity diagrams*, Cambridge Univ. Press, Cambridge, 1996.

[GS1] V. Guillemin and S. Sternberg, *Geometric asymptotics*, Amer. Math. Soc., Providence, RI, 1977.

[GS2] ———, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.

[H] Harish-Chandra, *Differential operators on a semisimple Lie algebra*, Amer. J. Math. **79** (1957), 87–120; Collected Papers, vols. I–IV, Springer-Verlag, Heidelberg, 1989.

[H^{2}SV] M. Hazewinkel, W. Hesselink, D. Siersma, and F. D. Veldkamp, *The ubiquity of Coxeter-Dynkin diagrams (an introduction to the $A - D - E$ problem)*, Nieuw Arch. Wisk. (3) **25** (1977), no. 3, 257–307.

[Hec] G. J. Heckmann, *Projections of orbits and asymptotic behavior of multiplicities for compact connected Lie groups*, Invent. Math. **67** (1983), 333–356.

[Hel] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, 1978, 1984; Corrected reprint of the 1984 original, Amer. Math. Soc., Providence, RI, 2000.

[Ho] R. Howe, *On Frobenius reciprocity for unipotent algebraic groups over $\mathbb{Q}$*, Amer. J. Math. **93** (1971), 163–172.

[Hu] J. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1972, 1980.

[I] I. M. Isaacs, *Characters of groups associated with finite algebras*, J. Algebra **177** (1995), 708–730; I. M. Isaacs and D. Karagueuzian, *Conjugacy in groups of upper triangular matrices*, J. Algebra **202** (1998), 704–711.

[J] N. Jacobson, *Lie algebras*, Interscience, New York, 1962; Dover, 1979.

[Ka1] V. G. Kac, *Infinite dimensional Lie algebras*, Cambridge Univ. Press, Cambridge, 1990.

[Ka2] ———, *Lie superalgebras*, Adv. Math. **26** (1977), 8–96.

[KP] V. Kac and D. H. Peterson, *Defining relations of certain infinite-dimensional groups*, Astérisque, Numero Hors Serie (1985), 165–208; *Regular functions on certain infinite-dimensional groups*, Arithmetic and Geometry, vol. II, Birkhäuser, Boston, 1985, pp. 141–166.

[KV] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff formula and invariant hyperfunctions*, Invent. Math. **47** (1978), 249–272.

[KO] B. Khesin and V. Yu. Ovsienko, *Symplectic leaves of the Gelfand-Dikii brackets and homotopy classes of nonflattening curves*, Funkcional. Anal. i Prilozhen. **24** (1990), no. 1, 38–47; English transl. in Functional Anal. Appl. **24** (1990).

[Ki1] A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspekhi Mat. Nauk **17** (1962), 57–110; English transl. in Russian Math. Surveys **17** (1962).

[Ki2] ———, *Elements of the theory of representations*, "Nauka", Moscow, 1972, 1978; English transl., Springer-Verlag, Berlin–Hiedelberg–New York, 1976.

[Ki3] ———, *Constructions of unitary irreducible representations of Lie groups*, Vestnik Mosk. Univ. Ser. I **25** (1970), 41–51. (Russian)

[Ki4] ———, *Local Lie algebras*, Uspekhi Mat. Nauk **31** (1976), 55–75; English transl. in Russian Math. Surveys **31** (1976).

[Ki5] ———, *Introduction to the theory of representations and noncommutative harmonic analysis*, Encyclopaedia of Mathematical Sciences, Noncommutative harmonic analysis. I, vol. 22, Springer-Verlag, Berlin–Hiedelberg–New York, 1994.

[Ki6] ———, *Introduction to the orbit method.* I, II, Contemp. Math, vol. 145, Amer. Math. Soc., Providence, RI, 1993.

[Ki7] ———, *Geometric quantization*, Encyclopaedia of Mathematical Sciences, Dynamical systems. IV, vol. 4, Springer-Verlag, Berlin–Hiedelberg–New York, 1990, pp. 137–172.

[Ki8] ———, *Invariant operators on geometric quantities*, Itogi Nauki i Techniki. Sovremennye Problemy Matematiki, vol. 16, VINITI, Moscow, 1980, pp. 3–29; English transl. in J. Soviet Math. **18** (1982), no. 1; *On invariant differential operations on geometric quantities*, Funkcional. Anal. i Prilozhen. **11** (1977), no. 1, 39–44; English transl. in Functional Anal. Appl.

[Ki9] ———, *Merits and demerits of the orbit method*, Bull. Amer. Math. Soc. **36** (1999), 433–488.

[Ki10] ———, *Variation on the triangular theme*, E. B. Dynkin Seminar on Lie Groups, Amer. Math. Soc. Transl. Ser. 2, vol. 169, Amer. Math. Soc., Providence, RI, 1995, pp. 43–73.

[Ki11] ———, *Characters of unitary representations of Lie groups: Reduction theorems*, Funkcional. Anal. i Prilozhen. **3** (1969), no. 1, 36–47; English transl. in Functional Anal. Appl. **3** (1969).

[Ki12] ———, *Infinite dimensional Lie groups: their orbits, invariants, and representations; the geometry of moments*, Lecture Notes in Math., vol. 970, Springer-Verlag, Berlin–Hiedelberg–New York, 1982, pp. 101–123.

[Ki13] ———, *Introduction to family algebras*, Moscow Math. J. **1** (2001), 49–63.

[KG] A. A. Kirillov and A. D. Gvishiani, *Theorems and problems in functional analysis*, "Nauka", Moscow, 1979, 1988; English transl., Springer-Verlag, Berlin–Hiedelberg–New York, 1982.

[KK] A. A. Kirillov and M. L. Kontsevich, *The growth of the Lie algebra generated by two generic vector fields on the line*, Vestnik Moskov. Univ., Ser. I Math. Mech. **1983**, no. 4, 15–20. (Russian)

[KKM] A. A. Kirillov, M. L. Kontsevich, and A. I. Molev, *Algebras of intermediate growth*, Preprint, vol. 39, Institute Appl. Math., Acad. Sci. SSSR, 1983.

[KM] A. A. Kirillov and A. I. Melnikov, *On a remarkable sequence of polynomials*, Algèbre non commutative, groupes quantiques et invariants (Reims, 1995), Soc. Math. France, Paris, 1997, pp. 35–42.

[KN] A. A. Kirillov and Yu. A. Neretin, *The manifold A_n of structure constants of n-dimensional Lie algebras*, Problems in Modern Analysis, Moscov. Gos. Univ., Moscow, 1984, pp. 42–56. (Russian)

[KY] A. A. Kirillov and D. V. Yuriev, *Representations of the Virasoro algebra by the orbit method*, J. Geom. Phys. **5** (1988), 351–363.

[Kl] W. H. Klink, *Nilpotent groups and anharmonic oscillators*, Noncompact Lie groups and some of their applications (E. A. Tanner and R. Wilson, eds.), Kluwer, Dordrecht, 1994, pp. 301–313; P. E. T. Jorgensen and W. H. Klink, *Spectral transform for the sub-Laplacian on the Heisenberg group*, J. Analyse Math. **50** (1988), 101–121.

[Kly] A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, Selecta Math. (N.S.) **4** (1998), no. 3, 419–445; *Vector bundles, linear representations, and spectral problems*, Proc. Internat. Congress Math. (Beijing, 2002), vol. II, Higher Ed. Press, Beijing, 2003, pp. 599–613.

[Kn] A. Knapp, *Representation theory of semisimple groups. An overview based on examples*, Princeton Univ. Press, Princeton, NJ, 1986.

[KT] A. Knutson and T. Tao, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products. I. Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999), no. 4, 1055–1090.

[KMS] I. Kolár , P. Michor, and J. Slovák, *Natural operations in differential geometry*, Springer-Verlag, Berlin–Hiedelberg–New York, 1993.

[Kon] M. Kontsevich, *Formality conjecture*, Deformation theory and symplectic geometry (Ascona, 1996), Math. Phys. Stud., vol. 20, Kluwer, Dordrecht, 1996, pp. 139–156; *Deformation quantization of algebraic varieties*, Lett. Math. Phys. **56** (2001), no. 3, 271–294.

[Ko1] B. Kostant, *Quantization and unitary representations*, Lecture Notes in Math., vol. 170, Springer-Verlag, Berlin–Hiedelberg–New York, 1970, pp. 87–208; *Graded manifolds, graded Lie theory and prequantization*, Lecture Notes in Math., vol. 570, Springer-Verlag, Berlin–Hiedelberg–New York, 1977, pp. 177–306.

[Ko2] ———, *The solution to a generalized Toda lattice and representation theory*, Adv. Math. **34** (1979), 195–338.

[La] S. Lang, $SL_2(\mathbf{R})$, Addison-Wesley, Reading, MA, 1975.

[Le] D. A. Leites, *Introduction to the theory of supermanifolds*, Uspekhi Mat. Nauk **35** (1980), 3–57; English transl. in Russian Math. Surveys **35** (1980).

[LL] H. Leptin and J. Ludwig, *Unitary representation theory of exponential Lie groups*, Walter de Gruyter, Berlin, 1994.

[LV] G. Lion and M. Vergne, *The Weil representation, Maslov index and Theta series*, Birkhäuser, Boston, 1980.

[Mac] G. W. Mackey, *Imprimitivity for representations locally compact groups,* I, Proc. Nat. Acad. Sci. USA **35** (1949), 537–545; *Unitary representations of group extensions,* I, Acta Math. **99** (1952), 265–311.

[M] S. Mac Lane, *Categories for the working mathematician*, Second Edition, Springer-Verlag, New York, 1998.

[Mn1] Yu. I. Manin, *Gauge fields and complex geometry*, Springer-Verlag, Berlin–Hiedelberg–New York, 1988.

[Mn2] ———, *New dimensions in geometry*, Uspekhi Mat. Nauk **39** (1984), no. 6, 47–73; Lecture Notes in Math., vol. 1111, Springer-Verlag, Berlin–Heidelberg–New York, 1985, pp. 59–101.

[MPR] W. G. McKay, J. Patera, and D. W. Rand, *Tables of representations of simple Lie algebras.* vol. I. *Exceptional simple Lie algebras*, CRM, Montreal, 1990.

[Mo] C. C. Moore, *Decomposition of unitary representations defined by a discrete subgroup of nilpotent groups*, Ann. of Math. (2) **82** (1965), 146–182.

[Nel] E. Nelson, *Analytic vectors*, Ann. of Math. (2) **70** (1959), 572–615; E. Nelson and W. Stinespring, *Representation of elliptic operators in an enveloping algebra*, Amer. J. Math. **81** (1959), 547–560.

[Ner1] Yu. A. Neretin, *Categories of symmetries and infinite-dimensional groups*, Oxford Univ. Press, Oxford, 1996.

[Ner2] ———, *An estimate for the number of parameters defining an n-dimensional algebra*, Math. URSS-Izv. **30** (1988), no. 2, 283–294.

[Ni] A. Nijenhuis, *Geometric aspects of formal differential operations on tensor fields*, Proc. Internat. Congress Math. (Edinburgh, 1958), Cambridge Univ. Press, New York, 1960, pp. 463–469.

[Ok] A. Okounkov, *Why would multiplicities be log-concave?*, The orbit method in geometry and physics (Marseille, 2000), Birkhuser, Boston, 2003, pp. 329–347.

[OPR] M. A. Olshanetskii, A. M. Perelomov, A. G. Reyman, and M. A. Semenov-Tian-Shanskii, *Integrable systems*. II, Encyclopaedia of Mathematical Sciences, Dynamical Systems. VII, vol. 16, Springer-Verlag, Berlin–Hiedelberg–New York, 1994, pp. 83–259.

[OV] A. Onishchik and E. Vinberg, *A seminar on Lie groups and algebraic groups*, Nauka, Moscow, 1988, 1995; English transl., Springer-Verlag, Berlin–Hiedelberg–New York, 1990.

[O] B. Orsted, *Induced representations and new proof of the imprimitivity theorem*, J. Funct. Anal. **31** (1979), 355–359.

[PT] R. Palais and C. L. Terng, *Natural bundles have finite order*, Topology **16** (1977), 271–277.

[PWZ] M. Petkovsek, H. Wilf, and D. Zeilberger, $A = B$, A. K. Peters, Wellesley, MA, 1996.

[PS] A. Presley and G. Segal, *Loop groups*, Clarendon Press, Oxford, 1986.

[Pr] R. Proctor, *Odd symplectic groups*, Invent. Math. **92** (1988), 307–332.

[P1] L. Pukanszky, *On the theory of exponential groups*, Trans. Amer. Math. Soc. **126** (1967), 487–507.

[P2] ———, *Unitary representations of solvable Lie groups*, Ann. Sci. École Norm. Sup. **IV** (1971), 457–608.

[QFS] P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, and E. Witten (eds.), *Quantum fields and strings: A course for mathematicians*, vols. I, II, Amer. Math. Soc., Providence, RI, 1999.

[RS] A. G. Reiman and M. A. Semenov-Tian-Shanskii, *Group-theoretic methods in the theory of integrable systems*, Encyclopaedia of Mathematical Sciences, vol. 16, Dynamical systems. VII, Springer-Verlag, Berlin–Hiedelberg–New York, 1994, pp. 116–262.

[R] R. W. Richardson, *Conjugacy classes in Lie algebras and algebraic groups*, Ann. of Math. (2) **86** (1967), 1–15.

[Ro] W. Rossmann, *Kirillov's character formula for reductive Lie groups*, Invent. Math. **48** (1971), 207–220.

[Sch] Y. A. Schouten, *Tensor analysis for physicists*, Clarendon Press, Oxford, 1959.

[Sh] I. M. Shchepochkina, *On representations of solvable Lie groups*, Funkcional. Anal. i Prilozhen. **11** (1977), no. 2, 93–94; English transl. in Functional Anal. Appl. **11** (1977); *Orbit method for the restriction-induction problems for normal subgroup of a solvable Lie group*, C. R. Acad. Bulgare Sci. **33** (1980), 1039–1042.

[So] Ya. Soibelman, *Orbit method for the algebra of functions on quantum groups and coherent states*. I, Int. Math. Res. Notices **6** (1993), 151–163.

[S] J.-M. Souriau, *Systemes dynamiques*, Dunod, Paris, 1970.

[T] C. L. Terng, *Natural vector bundles and natural differential operators*, Amer. Math. J. **100** (1978), 775–828.

[To] P. Torasso, *Quantification géométrique, opérateurs d'entrelacement et représentations unitaires de* $\widetilde{SL}_3(\mathbb{R})$, Acta Math. **150** (1983), 153–242.

[Veb] O. Veblen, *Differential invariants and geometry*, Atti Congr. Int. Mat. Bologna, 1928.

[Ver1] M. Vergne, *Construction de sous-algèbres subordonée a un element du dual d'une algèbre de Lie résoluble*, C. R. Acad. Sci. Paris **270** (1970), 173–175; 704–707.

[Ver2] ———, *On Rossman's character formula for discrete series*, Invent. Math. **54** (1979), 11–14.

[Vo1] D. Vogan, *Representations of real reductive groups*, Birkhäuser, Boston, 1981.

[Vo2] ———, *The orbit method and unitary representations of reductive Lie group*, Algebraic and Analytic Methods in Representation Theory (Sonderborg, 1994), Perspect. in Math., vol. 17, Academic Press, San Diego, CA, 1997, pp. 243–339.

[Wa] N. Wallach, *Real reductive groups*. I, Academic Press, 1988.

[We] A. Weinstein, *Lectures on symplectic manifolds*, CBMS Lecture Notes, vol. 29, Amer. Math. Soc., Providence, RI, 1977.

[Wh] H. Whitney, *The self-intersection of a smooth n-manifold in $2n$-space*, Ann. Math. **45** (1944), 220–246.

[Wi1] E. Witten, *Supersymmetry and Morse theory*, J. Diff. Geom. (1982), 661–692.

[Wi2] ———, *Coadjoint orbits of the Virasoro group*, Commun. Math. Phys. **114** (1988), 1–53.

[Y] K. Yosida, *Functional analysis*, Springer-Verlag, Berlin–Hiedelberg–New York, 1994.

[Za] D. Zagier, *Values of zeta-functions and their applications*, First European Congress Math. (Paris, 1992), vol. II, Birkhäuser, Basel, 1994, pp. 497–512.

[Ze] A. Zelevinsky, *Representation of finite classical groups*, Lecture Notes in Math., vol. 861, Springer-Verlag, Berlin–Hiedelberg–New York, 1981.

[Zh1] D. P. Zhelobenko, *Compact Lie groups and their representations*, "Nauka", Moscow, 1970; English transl., Amer. Math. Soc., Providence, RI, 1973.

[Zh2] ———, *Representations of reductive Lie algebras*, "Nauka", Moscow, 1994. (Russian)

Index

牛顿将其分析学中的发现用变位的形式进行了加密，破译后的句子是“It is worthwhile to solve differential equations”（解偏微分方程很重要）。因此，人们在表达轨道法背后的主要思想时可以说“It is worthwhile to study coadjoint orbits”（研究余伴随轨道很重要）。

轨道法由作者在 1960 年代引进，一直是诸多领域中十分有用和强大的工具，这些领域包括：李理论，群表示论，可积系统，复几何和辛几何，以及数学物理。本书向非专家描述了轨道法的要义，第一次系统、详细、自足地阐述了该方法。全书从一个方便的“用户指南”开始，并包含了大量例子。本书可以用作研究生课程的教材，适合非专家用作手册，也适合数学家和理论物理学家做研究时参考。

本版只限于中华人民共和国境内发行。本版经由美国数学会授权仅在中华人民共和国境内销售，不得出口。

美国数学会经典影印系列

AMS
AMERICAN MATHEMATICAL SOCIETY

这本书描述了各类李群的轨道法，是一本写得很好、系统化且可读性很强的书，是工作中使用表示论的数学家和理论物理学家的必备藏书！

—EMS Newsletter

ISBN 978-7-04-046910-3

9 787040 469103 >

定价 169.00 元